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12th February, 1925.

I ADDRESS myself to-night more particularly to those who are, or are to be, engaged in the design and construction of optical instruments and still more specially to the younger men on whom the future of the craft depends. I have a profound admiration for the knowledge, the skill, and the ingenuity of those to whom we owe the advancement of the theory of optics and of optical instruments, though I have not made the higher ranges of their work my special study. I include those of our Society who devote themselves to the mathematics and physics of optics among the workers in pure science to whom I have to refer, though their investigations may be specially directed towards the advancement of optics as an applied science.

The closing years of the 18th century and the early years of the 19th witnessed a great development in engineering practice on purely civilian undertakings and so much more method and science were then brought to bear on the design and execution of works of construction that civil engineering rapidly rose to the rank of a profession. To no one more than to John Smeaton was this due. James Watt called him "Father Smeaton," and said that "his example and precepts have made us all engineers." George Stephenson pronounced Smeaton to be "the engineer of the highest eminence that has yet appeared in England"; "Smeaton is the greatest philosopher in our profession that the country has yet produced"; "He was indeed a great man possessing a truly Baconian mind."

To a like extent James Watt brought to bear on the practice of mechanical engineering a mind trained to consider scientifically the problems that presented themselves in his practice. He was a master in scientific reasoning based on sound knowledge of the facts and laws of the physics and chemistry of his time. His many inventions contributed more than the work of any other man towards laying the foundation for that vast and rapid development of the mechanical arts that characterized the 19th century.

Smeaton—who may thus be ranked as the founder of the civil engineering profession—and Watt—who raised the practice of mechanics to so great a power for the advancement of civilization—were trained as scientific instrument-makers and not in the arts of construction, as practised in their early days, on works of a scale we usually associate with the term "engineering." Many others who have done large works in engineering have come from the ranks of your calling.

The profession to which I properly belong thus owes its present status largely to the infusion of skill and knowledge acquired in yours. The fact that you were kind enough, and may I say bold enough, to honour me with a call to the high position I now hold shows that you feel no resentment when men who have been trained as engineers exhibit in their careers a "reversion to type" in joining in your activities. Reversion in this case perhaps does not imply such a degradation from a higher to a lower state of development as is usually associated with the expression in regard to matters biological. One may safely assume that you, at any rate, do not find in such migrations as those I have referred to, illustrations of Tennyson's vision of

> Evolution ever climbing after some ideal good
> And Reversion ever dragging Evolution in the mud.

But, after all, there is no difference in type between the engineer and the scientific instrument-maker. Both depend for success upon the same *ingenium* from which the grosser branch derives its designation (though it is only north of the Tweed that the proper initial sound is still retained in the pronunciation of the title); and there being no barrier between the practice of the engineer and that of the instrument-maker, wanderings across the ill-marked frontier are natural, and may in some cases be beneficial even when the trespass takes the reverse direction from that which I have cited as being so fruitful in great results. I do not mean to suggest that the engineer who adopts the profession or the craft of the optician is likely to initiate any great revolution in the methods or the scope of your activities, but men who have no bias founded upon long practice in a particular branch of a craft as it has been and is pursued, are naturally more disposed to consider how a thing might or ought to be done than those who have much experience and skill in one way of doing it.

It is told that, when the Temple was nearing completion, Solomon invited the chief craftsmen to a banquet. Conversing with his guests, he asked one man to what craft he belonged. "I am a carpenter." "And who makes your tools?" "The blacksmith." "And what are you?" he asked another. "I am a stonecutter." "And who makes the tools you use?" "The blacksmith makes my tools." "And your calling?" "I am a builder." "And who makes the appliances you use?" "The blacksmith." Coming at last to a man of powerful build and sinewy arms, he asked what he was. "I am the blacksmith." "And who makes your tools?" "I make my own tools." The man of wisdom pronounced the blacksmith the king of the craftsmen because he not only made the tools he used in his own craft but those upon which the other crafts depended.

A parallel is not difficult to draw. In preparing the elements that go to the building of the "Temple of Knowledge," the scientific instrument-maker, and

especially the optical instrument-maker, more than any other, is the tool maker for all the branches of scientific investigation, including his own: but the optician who would, on that ground, claim kingship over all other workers in science could hardly be credited with the wisdom of a Solomon. With less egotism I may draw the parallel, if not pronounce the decree, being only a partially naturalized alien in your special province, with little hold on your traditions, an imperfect command of your dialect, and no claim to share in your birthright.

When we review the processes by which science is advanced, we cannot fail to be impressed with the extent to which the investigators, in almost all branches, depend upon the knowledge and skill of the optician for the provision of the tools that enable them to carry on their work. You provide the implements by which the astronomer measures up the heavens and those by which the geodesist maps out the earth's surface. The petrologist owes to you the microscope which reveals to him the structure of the materials of which the earth's crust is made up and the processes by which they have been produced. We would know nothing of the constitution of the sun and the stars had not the spectroscope been at the service of the astrophysicist and indeed given birth to his science. The botanist and zoologist of to-day find the microscope their chief tool and so do the histologist, the pathologist, and the physiologist in some branches of his work. There would be no knowledge of bacteriology had the microscope not been at hand ready to reveal many of the secrets of disease and of processes upon which life itself depends. And optical instruments now play a most important part in many investigations concerning the practical arts.

The survey of the use made in investigation of the tools you produce might be indefinitely extended, and one must not forget that, from the philosopher to the man in the street, many of us are almost entirely dependent for such efficiency as we can claim in the discharge of our daily duties to the aid you give in overcoming the limitations and defects of our unaided vision.

No doubt it would be true, in a superficial sense, to say that the brush and the chisel are the tools of the painter and the sculptor; but the perfection of such tools has little to do with either the extent or the quality of the artist's output. It is very different in the case of the tools you provide. Every improvement you make in the products of your science and skill enables the user of these tools to go farther and deeper in his researches. Were you to provide to-morrow telescopes or microscopes of much greater power than the best hitherto available, a flood of new discoveries would be announced the day after. Sir Humphry Davy—himself not only a great investigator, but a great natural philosopher—is quoted as having said:

> Nothing tends so much to the advancement of knowledge as the application of a new implement. The native intellectual powers of men in different times are not so much the cause of the different success of their labours as the peculiar nature of the means and artificial resources in their possession.

You may, therefore, have a share, and in some cases a large share, in the credit for new advances in science in departments of research far removed from your own.

The whole may be summed up in the reflection that, of the five senses, vision—

to the almost complete exclusion of the others—is the faculty upon which, directly or indirectly, all scientific work, and very much of our ordinary occupations, depend, and your function is to provide more and more delicate and powerful means for increasing its effectiveness.

In what I have said of the great part your products play in almost all branches of research it may appear that I am giving all the credit to the professional optician and the optical craftsman. Let us remember, and most candidly confess, that in many, if not indeed in most of the important path-breakings, the credit for the invention of the tools—and in not a few cases also for their fabrication—is due to men of science who had not made the practice of optics their special vocation; many of the greatest advances in optical practice have not originated in the optical workshop. Without the great skill of the optical designer and craftsman instruments of the necessary refinement could not have been effectively produced, but their inception and much of their design in regard to mode of operation at least has come from men of other callings who felt the need for them in carrying out some special work of investigation.

I do not propose to enlarge upon this subject in its present application; I cannot claim to be an authority on the history of optical inventions; those who have not made a profound study of all that has led up to an embodiment of slowly accumulated knowledge of principles and of data in a device are too apt to give undue credit to him who made the last of the long series of forward movements by which an end has been reached.

Lowell said of aphorisms:

> Though old the thought and oft expresst,
> 'Tis his at last who says it best.

These lines themselves are a most apt illustration of the truth they state, as (perhaps 150 years earlier) Pope had said:

> True wit is Nature to advantage dress'd,
> What oft was thought but ne'er so well expressed.

And so it is with inventions. The true origin of a device is seldom indicated by the personal name with which it comes to be associated. We should never forget that, however novel a product of our ingenuity may appear to be, at best it embodies

> The wisdom of many and the wit of one.

In the progress from crude beginnings of any craft that depends upon the application of scientific principles, great advances can be made only by men who can look behind and below the practice of their time, and in the earlier stages such men are usually found outside the craft itself. If we study the development of the iron and steel industry, for example—say from the beginning of the 18th century to the middle of the 19th—and examine the careers of the men who have introduced the greatest and most fruitful departures from the then current practice—such men as Roebuck, Cort, Huntsmann, Mushet, Neilson, Bessemer, Siemen, Snellus, Thomas and Gilchrist—we find that not one of them was trained in the practice of iron and steel manufacture as carried on before they intruded their new ideas

upon the industry—unless we except Mushet, who was an accountant, and Snellus and Gilchrist, who were chemists in iron works. So with the building of lighthouses. The first Eddystone lighthouse was built of timber by a London mercer—he lost his life when it was wrecked—the second, again of timber, was designed by another London mercer, and when it was burned down, no architect or craftsman in wood or stone was considered competent, or was sure enough of his skill, to replace it by a more permanent structure. It was the then President of the Royal Society who recommended the promoters to entrust the design to Smeaton, who, as we have said, was trained as a scientific instrument-maker. He adopted stone as his material, and very novel and hitherto untried modes of construction, with admirable results.

Since the formulation of the principles of applied science, in regard to works of construction and processes of manufacture, the case is very different. Among those engaged in most of the practical arts there are now men who have the requisite scientific knowledge combined with experience, and advances usually originate within the industry itself. In practical optics the need for sound scientific knowledge has naturally been realized to a larger extent than in the less evidently scientific industries; but it is only of recent years that there has been any considerable body of men familiar alike with the craft and the science of your calling. For the future there is no excuse for ignorance on the part of those engaged in practical optics, either of the principles or of the data upon which advance must depend, and this Society is doing a great work in formulating and diffusing such knowledge. Still in the future, as in the past, many of the most important advances will originate with investigators who, from the trend of the researches in which they are engaged, foresee what further aids they will require and, with a unique knowledge of the end to be attained, and a general knowledge of the principles that may be applied, can themselves devise the appliances that will best meet their requirements. It is up to the opticians of to-day—or at least the few who have the ability, and the perseverance necessary for following the latest advances in science—to keep themselves familiar with the main lines of advance, to understand the requirements that arise, so as to be ready to forestall the needs of the investigators, and to bring to bear on the production of the new tools an intimate knowledge of the means by which, and the extent to which, these requirements can be fulfilled in a practicable device.

As most of you know, the optical work on which I have been engaged has for the greater part not been connected with tool-making for investigators in pure science. I do not expect—or for your own sakes hope—that many of you will be troubled with the special problems of rangefinder design, but some of the devices that have been developed in the solution of these problems may be suggestive of methods of operation or details of construction that could be applied with advantage in instruments of quite a different type and for very different purposes.

In this connexion I may remark that it is sometimes an advantage for a would-be inventor not to know what has been previously done towards solving the particular problem he proposes to attack. If he does know what has already been done he

is very apt at first to be discouraged because he thinks the problem has already been considered and so far solved by others more able than himself, or, in regard to details, he is disposed to accept a mode of construction or of operation that has already been used as the best means of accomplishing his end.

It is much easier, and it seems safer, to follow a beaten track than to blaze a new trail. A footpath becomes widened out and consolidated or deliberately formed into a road; but its trend may, from a want of knowledge or foresight, not have started off in the best possible direction. Part of the route may have been chosen when the destination was not quite that we now wish to reach, and with new means of transport the best set out may now be in some quite different azimuth. On the other hand familiarity with successful trekking that has been done in other—perhaps distant—territories may suggest the best path to break in this, if one is not led astray by the evidence before him of the way others have taken in their endeavours to reach his goal or one near to it. So it is in invention. A familiarity with many devices that have been successfully adopted towards other ends is of the greatest value to the inventor. When a new problem presents itself, a variety of devices come naturally to his recollection and suggest many different means that he may choose among for accomplishing his end, whereas, if he has uppermost in his mind one tried method, he is very apt to accept it as so far showing the course he should follow.

However far this may or may not be true, I think it was an advantage to Dr Stroud and myself that when, in 1888, we began the design of a short base rangefinder to fulfil the requirements specified in a War Office advertisement, we were not aware that anyone had previously considered the problem. It was not until after we had settled the general principles, the main outlines, and many of the details of our rangefinders that we looked into the Patent records and found that Adie in 1860, Mallock in 1885, and Christie in 1886 had devised self-contained base telemeters. We were somewhat surprised to find that some of the general features that we had invented for ourselves were not new. Had we known that men of such eminence had already attacked the problem we might not have considered it worth our while to enter the field.

In what I have to say of rangefinder design there is nothing that might be referred to as belonging to higher optics. The problems comprise only questions of a simple optical kind. The special features of the problems are that simple elements are to be combined in an instrument that will work to a great degree of accuracy, be very easy for an ordinary soldier or sailor to use, and at the same time be robust enough to stand usage of a roughness that fortunately no other instrument of precision has to endure.

The treatment to which infantry and artillery rangefinders may be subjected in the field can be better imagined than described, and naval rangefinders, which have to work to an accuracy of a fraction of a second of angle, have to withstand the shocks of gun fire, and such changes of temperature as are met with from the northern seas to the tropics.

Many of you are no doubt familiar with the general principles of construction

of self-contained base rangefinders working on what is now called the coincidence principle and of others working on the stereoscopic principle. Such instruments are usually called short-base rangefinders though they have been constructed with bases from 50 cm. to 100 feet.

Of stereoscopic rangefinders I do not propose to speak at length. This type was adopted to a large extent in Germany but the newer German rangefinders are in many cases—if not in most—constructed on the coincidence principle. Barr and Stroud have made instruments of the stereoscopic type when such were demanded but the system is now largely abandoned as the result of experience. Our own tests and those of others indicate that with good observers in each case a rangefinder of the coincidence type is about twice as accurate as one of the stereoscopic type of the same dimensions. But more important even than the question of the accuracy obtainable under the best conditions there is the fact that only about one observer in twenty can be trained to work the stereoscopic type with reasonable accuracy and certainty. Again, even if the observer is thoroughly qualified and trained, a derangement of his power of stereoscopic observation is readily produced by a slight indisposition, by fatigue or by the unfavourable conditions under which he has to work, and the conditions must always be unfavourable in the field or on board a battleship during an action. With even slight derangement of the stereoscopic sense the observations may be wildly wrong.

Tennyson truly said:

> Sometimes the small black fly upon the pane
> May seem the black ox of the distant plain.

Stripped of symbolism these lines indicate that even a poet's sense of stereoscopic vision is sometimes sadly upset by, shall we say, a slight bilious attack.

In coincidence rangefinders two partial images of the object viewed are seen either superposed upon one another or one above the other separated by a more or less fine line of demarcation called the separating line (see Figs. 8, 9, 10). If the four mirrors represented in Fig. 1 were set exactly at 45° to the axis of the base

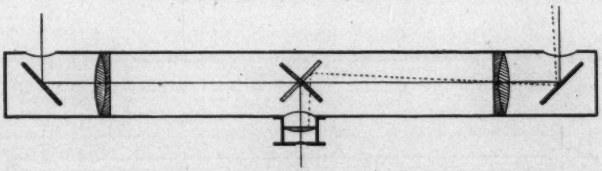


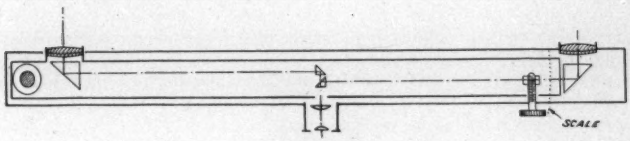
Fig. 1

tube and the optical centres of the objectives and the crossing of the central mirrors were on the axis, an object at infinity would be represented by two images in alignment or coincidence with each other. If the object now comes nearer, say directly towards the left-hand reflector, the axis of the right-hand beam will, after reflection, make a small angle with the axis of the instrument and the corresponding

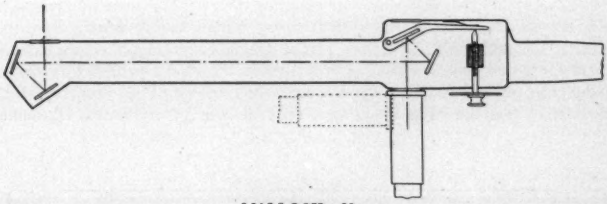
image will fall to the right of the other. The relative displacement of the images might be measured to determine the range but this would be a very inconvenient and inaccurate operation. By some mechanical or optical means one of the images is moved so as to come into alignment with the other, and the displacement of the mechanism from its infinity position required to effect coincidence indicates the range. One of the chief features in which the various instruments that have been devised differ from each other is in the means adopted for bringing the partial images into coincidence or alignment.

There are at least six more or less simple methods that have been used for this purpose.

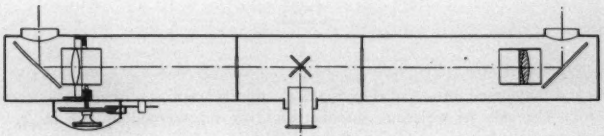
(*a*) Angling one of the end reflectors. This was used by Adie in his rangefinder (Fig. 2) of 1860 and also by Mallock in 1885 who angled one of a pair of mirrors forming an optical square (Fig. 3).



ADIE, 1860.
Fig. 2



MALLOCK, 1885.
Fig. 3



CHRISTIE, 1886.
Fig. 4

(*b*) One of the objectives may be translated in the plane of triangulation in a direction normal to the base, carrying the image with it. This was the method adopted by Christie (the late Astronomer Royal) in his rangefinder (Fig. 4) of 1886.

(*c*) A parallel plate of glass may be introduced into one of the convergent beams and moved from its normal attitude so as to translate the beam. This we have used for making adjustments but I am not aware that it has been adopted as the means of operation in range-taking.

(*d*) Two prisms may be introduced in one of the beams either in front of one of the objectives or between the objective and the central reflector and rotated relatively to each other so as to alter the angle of deviation in the plane of triangulation. This method was adopted for special reasons in one or two rangefinders made by us. It is (or was) the method usually used by German firms.

(*e*) A deviating prism may be inserted in one of the beams and angled from its position of minimum deviation. This deviation was used by James Watt* in a telemeter (which he called a micrometer) devised for surveying purposes (Fig. 5). This method was used by Cooke.

"Another micrometer, with a prism, I invented, I think, about that time. It consisted of a thin prism almost parallel (say of one degree or two). This prism was cut by a diamond into two parts, which, when they were fixed in the same plane, refracted all the rays which passed through them equally; but one of them remaining fixed, and the other moving on a centre at *a* (according to the pricked line), would refract that portion of the rays which passed through it more than those which passed through the fixed part; and being fixed in the focus of the object-glass of a telescope, two images were formed of each object, by which its diameter could be measured. An index and divided sector of a circle served to measure the comparative refractions. This instrument I made with the sector and

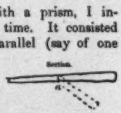


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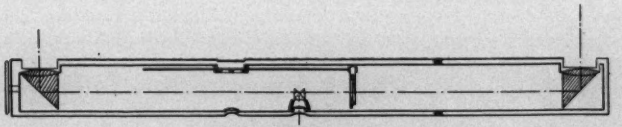
Fig. 5

(*f*) A deflecting-prism may be inserted in one of the convergent beams and translated along the beam, as in the Barr and Stroud rangefinders (Fig. 6).



BARR & STROUD, 1888.
Fig. 6

* Muirhead, *The Origin and Progress of the Mechanical Inventions of James Watt*, vol. I, p. 135.

These methods are not stated in historical order nor is the list intended to be complete.

When Dr Stroud and I started devising a rangefinder—in 1888—after considering means for angling one of the end reflectors and realizing the difficulty of moving a mirror or prism very delicately, steadily, and accurately, we adopted the translating deflecting-prism method (Fig. 7) after the first few days of study of the problem, and we have not seen reason to depart from it in these 37 years, except in the case of one or two instruments in which method (*d*) was adopted for special reasons.

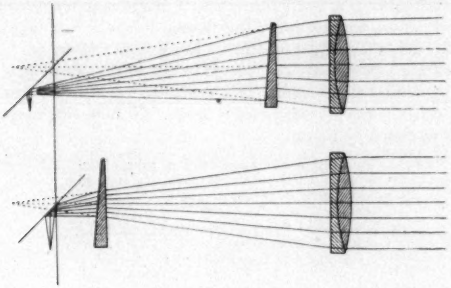


Fig. 7

Time will not permit of a comparison in detail of these methods as regards their relative advantages and disadvantages, but I may point out what seem to us to be advantages of the method we adopted.

In the first place it may be well to have some concrete idea of the problem from a quantitative point of view. I shall take this 80 cm. base instrument for purposes of illustration. It is intended to read ranges from infinity down to 250 yards with an accuracy of about 5 yards at 1000 yards or, otherwise put, of $\frac{1}{2}$ per cent. per 1000 yards of range—since the uncertainty of observation in all constant base rangefinders varies as the square of the distance. If the range be 250 yards the beams entering the instrument will be inclined to each other at an angle of $\frac{B}{R}$ where B is equal to 80 cm. and R is equal to 250 yards. This angle is about 12 minutes and taking the focal length of the objective as 11 inches the corresponding relative movement of the partial images in the focal plane is about $\frac{1}{20}$ inch or 1 mm. This is the total angle—or linear movement—within which the whole scale of ranges has to be got. If the instrument is to read to $\frac{1}{2}$ per cent. per 1000 yards, that practically means that this angle—or length—has to be divided into 800 equal parts. The angle between the beams has thus to be measured to about 0·9 second or the relative linear movement of the image to $\frac{1}{20000}$ inch.

A 5-inch theodolite is usually graduated to read by vernier to 1 minute. To read

to 0·9 second with equal facility would require a circle of about 30 feet diameter. But even in that case the scale would be altogether too difficult to read under rangefinding conditions.

It will be seen, therefore, that with method (*a*) one end reflector will have to be moved through a total angle of 6 minutes correctly to less than $\frac{1}{2}$ second, since the angle is doubled by reflection. By the method (*b*) the objective will have to be translated to an accuracy of less than $\frac{1}{200000}$ inch. By the other methods (*c*), (*d*), (*e*) or (*f*) a much larger angular or linear motion of the working part is secured.

The deflecting prism method has in the first place the advantage that the prism, if made of a suitably small angle of deviation, will have to be translated through a considerable fraction of the focal length of the objective for a change of range from ∞ to 250 yards. In this instrument the translation is 6·8 inches, so that we have that motion of the prism to produce a motion of the image of $\frac{1}{36}$ inch. The motion, therefore, does not need to be produced by any delicate micrometer system. Further, since a straight scale can be attached directly to the carrier of the deviating prism, there is no lost motion between it and the prism. Again, the effect of lost motion or irregularity of the gear driving the deflecting prism is altogether eliminated, since even if the gear used for moving the deflecting prism has slackness and a small motion of the working head fails to move the prism, it also fails to move the scale, and friction and elasticity of the parts and want of truth of the driving mechanism have practically no effect.

It may be here remarked that an optical mechanism of this kind (Fig. 7) is in a sense a perfect mechanism, in respect that a millionth of an inch or a millionth of a millimetre of translation of the prism must produce an exactly corresponding motion of the image assuming that the prism is not rotated about an axis parallel or perpendicular to the plane of deviation and as the prism is set at its angle of minimum deviation, and its deviation angle is small, it is comparatively insensitive as regards such motions and it can be readily guided with sufficient accuracy. This is not the case with any mechanical micrometer method in which lost motion, elastic yielding, want of accuracy, or wear of the screw—or other micrometer gear—affects the result.

If I may make a digression, I would sum up a great deal of my experience in engineering and in instrument design and construction in the statement that the function of a works manager and his workmen is to turn out the most accurate and sound work that can be produced within the limits of cost that the circumstances allow: the skill of the inventor and the designer on the other hand should be directed towards the elimination, as far as possible, of the need of accurate and difficult craftsmanship, the effects of qualities of the materials available, and the wear and tear on the finished product. In his admirable Presidential Address*, Mr Whipple gave many illustrations of the application of what Lord Kelvin used to call "geometrical control" in mechanisms and the treatment of the subject, both in that Address and in the paper by Clerk-Maxwell, to which Mr Whipple refers, should be very carefully studied by all designers of mechanisms. But we must remember

* *Trans. Opt. Soc.* 22 (1920–21) 35.

at the same time that there is often a great difference in the requirements between instruments that are to be used in the laboratory and those that have to withstand rough usage, and we have often in engineering structures and machines and also in some details of instruments such as rangefinders which are to be subjected to rough usage, to take advantage of the elasticity of materials to counteract in some measure departures from ideal geometric arrangements of mechanism.

In some forms of short-base rangefinders—in the Mallock instrument, for example—the two fields overlap (Fig. 8); that is to say, the images are ghosts, and in the operation of the instrument these ghosts are brought to superposition. The Mallock naval rangefinder was really a vertical base rangefinder but it has been drawn horizontally in Fig. 3 for comparison with the others. It therefore did not work on a mast as represented in Fig. 8, but on some point of the target having horizontal lines and the relative displacement was of course vertical. In other cases, in the Adie and Christie rangefinders, for example, the two images were separated

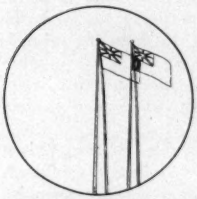


Fig. 8

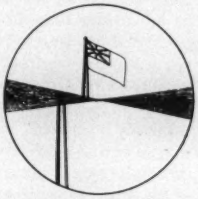


Fig. 9

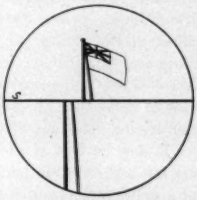


Fig. 10

by a more or less roughly marked line of division (Fig. 9). The optical system by which the partial images are now separated by a fine line in focus all across the field (Fig. 10) may be illustrated by a reference to one of the earliest forms of "eyepiece prisms" that we adopted; this system is easier to represent diagrammatically than any of the later forms. The particular form of eyepiece prisms represented in Fig. 11 was used in rangefinders with single reflecting mirrors or prisms at the end of the base, and an inspection of the diagram will show that this system turns the image right for right and re-inverts it, besides providing a fine separating edge. (Most instruments now have double reflection—that is to say, they have an optical square—at each end of the base. This feature was introduced first in the Mallock instrument.

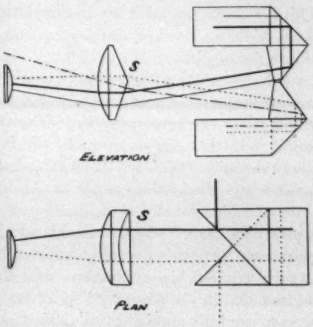


Fig. 11

In that case the eyepiece prisms may be designed to re-invert the images but do not require to reverse them about a vertical line.)

It will be evident from the diagram that all of the one image that falls above the edge *S* of the separating prism is brought by refraction horizontally into the eyepiece while the rays forming the remainder of that image are *wandered* by passing through a prism of contrary deviation; in other words, each image is separated into a visible and an invisible part. The fineness of the line of separation depends upon the accuracy with which the edge of the prism is made fine, true, and free from chippiness. It is found by experiment that a very fine line of separation is necessary in order to get the best determination of accurate coincidence.

It may be remarked in passing that in some rangefinders—especially in instruments for use in the field—one of the partial images is seen inverted—usually the upper one.

But, while a fine line of separation is required for the greatest accuracy, an indefinitely fine line (that is to say an invisible line) would not do, because the eye would not then be guided to the particular spot on the image at which a want of coincidence is to be looked for.

Now it will be seen from Fig. 11 that, as no light approaches the separating prism in a horizontal direction, a facet, or a rounding of the edge, produces an absolutely black line across which the coincidence is to be observed. The skill of the workman is exercised in making the prism free from chippiness and scratches and with so small a rounding as to make the line as narrow as possible, consistent with visibility under the eyepiece to be used. This is no doubt one of the most difficult parts of the instrument to prepare with the required refinement. I might here point out that, if such a separating prism is to be used in observations on a bright object on a dark background, a star, for example, light may be provided opposite the edge of the prism in which case the separating line will appear as a fine bright line upon the dark background.

The third feature I would refer to is the introduction of astigmatizing lenses (Fig. 12) to elongate the images of any small object, say, for example, the image of

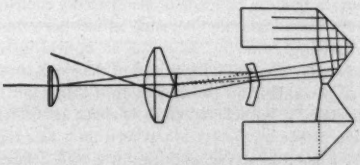


Fig. 12

a star used in testing the instrument for infinity adjustment. The cylindrical lenses may be introduced at any place along the beam before it reaches the separating line. They are sometimes, as in Fig. 12, inserted between the objectives and the separating prism and sometimes outside the objectives. If the astigmatic lenses

are of suitable strength, in relation to their distance from the image plane, a star gives a most beautifully clear vertical line (Fig. 13) and a lighthouse or a brightly illuminated object gives a series of bright vertical lines (Fig. 14). By this means a point of light can be worked upon as exactly as a mast or pole can be worked upon in daylight, and small objects seen in daylight can also be converted into a combination of brighter and darker clearly defined vertical bands.

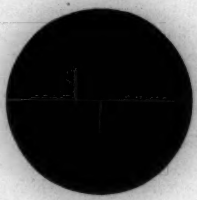


Fig. 13

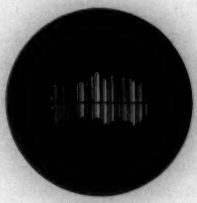


Fig. 14

In connexion with the method of observing by coincidence across a fine separating line, the question of the limit of resolving power of the telescope is often raised, but really the resolving power, as usually understood, does not apply. In this 80 cm. base rangefinder the effective width of the objectives is about 1 inch. The ordinary calculation would show that the resolving power of the telescope is about 5 seconds, but a skilled operator can work far within that limit. Not only so, but it has been found that there is little loss of accuracy, if any, when the objectives are stopped down to a width of $\frac{1}{8}$ inch or less, if the object be a bright line on a dark background. Of course there is considerable diffraction at the edges of each image but the diffracted images can be brought into alignment with practically the same accuracy as if no visible diffraction is present, and indeed the diffraction bands themselves may be aligned. The "resolving power" of a telescope of $\frac{1}{8}$ inch aperture is 40 seconds; still, observation can be made within, let us say, 1 second, neglecting the much finer observations that can be made under specially controlled circumstances.

Again, with a very fine separating line, such as has been described, it is found that the limit of resolving power *of the eye* as an optical instrument, as usually stated, seems not to apply. The resolving power of the eye is sometimes given theoretically as about 40 seconds from consideration of diffraction, and the rods and cones of the eye are stated to subtend an angle of about 40 seconds also. *Practically* the limit of resolving power by the eye is put at from 1 to 2 minutes. It is found quite regularly that a good observer working on a well-defined vertical line can set coincidence to within a small fraction of this angle at the eye.

This feature of the coincidence method is very fully treated in Dr French's paper on "The Unaided Eye," read before this Society*, and in the discussion which followed its presentation, and various theories are advanced to account for it. We

* *Trans. Opt. Soc.* 21 (1919–20) 127.

early explained it to ourselves much on the lines of the explanations very clearly stated by Mr Smith and Dr Clay in the discussion.

I shall only give one illustration of what I have referred to as the possibility of features developed in rangefinder design being used in instruments for quite other services. For some astronomical purposes an instrument might be constructed more or less on the lines of an instrument for surveying purposes described in a patent we took out in 1889 (no. 11,025). The instrument consists of a telescope with a deflecting prism introduced into one half of the beam of the objective or preferably reversed deflecting prisms in the two halves. Two targets at a fixed distance apart are carried upon a pole which is held up at the point whose distance is required. The images of the targets are brought into coincidence by movement of the deflecting prism and the distance of the staff is thus determined. I would ask, if an instrument more or less on these lines, fitted with *astigmatizers and separating prism*, could be usefully employed in astronomy, for example, for the measurement of the angular distance between stars, and more especially in the measurement of the separation of the two elements of a binary, in cases in which the members are sufficiently separated to be resolvable by the telescope, so as to appear as two vertical bright lines in each partial field. Possibly it would be best to have two objectives rather than one objective half of which is used in the formation of each image. If the objectives were 10 cm. in diameter, all double stars having more than $1\frac{1}{4}$ seconds of separation would appear in each partial field as a pair of vertical lines. Under observatory conditions determination could probably be made within two seconds at the eye—perhaps much more finely, as Dr French has shown in the paper referred to*. But taking the determination as within 2 seconds at the eye and the magnifying power of the instrument as 100, the separation of the stars could be measured to, say, 0·02 second. I believe more accurate measurements would be possible with quite realizable refinements of construction. If the two members of the binary are of very different magnitudes one of the partial fields could be dulled down when the bright member in it is to be brought into coincidence with the fainter image in the other. Three observations could, of course, be made:

(1) double coincidence,
(2) upper field to the right,
(3) upper field to the left.

All these observations would be made by the same observer under like conditions and any bias in setting coincidence would be eliminated.

It will be noticed that such an instrument could be turned with its plane of separation parallel to the line joining the two stars and so the actual separation obtained directly. As in a rangefinder, perfect steadiness of the images in the field would not be necessary. Such an instrument could readily be constructed; it is for astronomers to say whether or not it would probably serve any useful purpose.

Optics is a part of the science of radiology which has been so surprisingly extended in recent years along the wide range of vibrations from beyond those of

* *Trans. Opt. Soc.* 21 (1919–20) 127.

X-rays to those of the long waves with which wireless telegraphy and telephony deal. I suppose there must be an upper limit to the periodicity of vibrations that are generated in the aether. I do not know what physicists have to say on that point. There seems to be no limit that could be set at the other end of the scale, since we can produce alternating currents of as slow a period as we please. But the radiations that are known and referred to as such, and have been more or less investigated, and in many cases used in other scientific investigations and in applied science, extend over more than sixty octaves. Opticians are primarily concerned with less than one octave out of the sixty or more, though for some purposes they have to take account of one or two octaves on each side of that of visibility into the infra-red and the ultra-violet. How far you will be called upon to go cannot be foreseen, but it is certain that in regard to the shorter waves, i.e. the high periodicity radiations, the materials used and the methods at present employed in the construction of optical instruments cannot be applied. Were it the case—which it is not—that the same type of appliance could be used for high period rays as is found suitable for work within the visible range, the workmanship would require to be of an entirely different order of refinement. A surface that would be considered perfectly smooth and flat for visual rays may be as rough, in comparison with one alike suitable for dealing with X-rays, as the Atlantic in a gale is rough compared with the sheet of ice on a skating pond, and a diffraction grating to deal with X-rays would require some 5000 lines in each space of a good grating for visual rays—if it were possible to have a smooth enough surface and appliances refined enough to draw them.

The investigations and discoveries that have chiefly characterized the last twenty-five years of progress in physics have for this reason depended less perhaps on the apparatus produced in the optician's workshop than the investigations in almost any other branch of natural science. Sight is still the faculty by which the phenomena are revealed and studied, but the processes by which the phenomena are brought within the range of visible indications are not such as the optician deals with. Photography—as in the case of the X-rays—is fortunately available for recording the paths of such radiations but, as it stands to-day, X-ray photography is in much the same stage of development as photography by light was in its most primitive form. Indeed the very earliest form of photography was able to record, for example, the structure of a leaf, by the shadow method, with a far greater degree of exactitude and refinement than the most advanced X-ray photography of to-day, incalculable as have been the benefits it has conferred in surgical and medical diagnosis. It may be remarked that we refer to the production of impressions on chemically sensitive plates as "photography," even though the rays employed are far removed from those of visible light. Will instruments akin in delicacy and complexity to the optical instruments of to-day be devised to deal with the finer radiations and will an extension of the etymological meaning be given to the term "optics" and "optician" such as has, naturally enough, been given to the word "photography"? Perhaps such instruments may be devised, but it is certain that, if so, they will be very different and depend on entirely

different principles and make use of different materials from those you at present apply and employ. Still, those who have mastered the science and art and craftsmanship of optical instrument design and construction may be better qualified than any others to launch out into such a new field of service to science. There seems, however, no need to extend the range of subjects dealt with in this Society. Our *Transactions* are growing in complexity and in the mass of detailed work they record with quite sufficient rapidity, and the one octave of the visible spectrum has and always will have a very special significance in the scheme of things as they are, narrow as it is on the scale of radiology. Apart altogether from possible extension of your sphere of activity, there is still, no doubt of it, scope for development within the range of optics, properly so-called, to which we can set no limits.

I saw in a newspaper a few days ago what purported to be the statements of a very eminent anthropologist to the effect that "Man's brain through the ages has been getting gradually smaller." "We have grown up upon the accumulated knowledge of our forefathers. There are fewer fresh problems for the brain to tackle, and the brain has consequently dwindled in size according to its needs." We must take the authority of specialists as regards scientific facts, but we are not to the same extent bound to agree with them as to inferences drawn from these facts. But I take the words as a text, not as a subject for criticism. In matters of heredity we are no doubt as a race concerned with the cumulative or averaged influence of our ancestry and there may be too much truth in the statement that men, in general, "have grown up upon the accumulated knowledge of their forefathers," and have rested more or less content with what they have inherited; and it may be true that by insufficient exercise they have allowed the brains of the race to atrophy in generation after generation. But one can hardly agree with what is given as the primary cause of this as of general application that there "are fewer problems to tackle." It is not so in the realm of science, pure or applied, at any rate. So far as we can see there is no limit to what Newton visioned as the "great ocean of truth"; the horizon ever recedes as we advance, and there is no assignable limit to the application of science for the "use and convenience of man."

In science and in the arts in general, if man is content to grow up merely on the accumulated knowledge of his forefathers, his own brain will not develop in quality whatever happens as regards dimension. The brain grows in effectiveness as it is exercised creatively and not simply receptively, and, as the circle of any branch of knowledge is widened, there are more points of departure for exploration and advance. No one can have lived during the last fifty years in more or less direct contact with those who have been working in pure and applied science without realizing that the number of men engaged in important researches has greatly increased, and the output of original and detailed work to-day is out of all comparison with that of half a century ago. There were giants in those days; but they were, as giants have always been, few and far between. But there are giants to-day and a far greater number of efficient workers of lower stature but with effective equipment, in knowledge and in opportunities, than ever before.

In optics, the telescope, the microscope, the spectroscope, and the interfero-

meter, to name only a few of the important appliances with which this Society has to deal, are already known and developed to great refinement. Are there on that account fewer fresh problems to be solved? Far from it. If the younger men of the Society tackle the problems of to-day with energy and with a sound knowledge of the scientific discoveries that may be applied in their solution—and, let me add, a profound conviction that advancement, great advancement, may be and will be made—they need have little fear of atrophy of their own brains from disuse: they will be in much greater danger, individually, of some day exhibiting a seemingly contrary symptom.

THE GEOMETRICAL SOLUTION OF COLOUR MIXTURE PROBLEMS

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ABSTRACT. Starting from the experimental fact that any colour can be uniquely expressed by a trichromatic equation, provided negative coefficients may enter, it is shown that all problems of colour mixture are amenable to an exact system of geometrical calculation. The particular methods described in the paper are such as to obviate the introduction of stereographic projection and other geometrical complications which enter into previously published methods. This simplification is effected by conducting the actual colour mixture part of any calculation *in the quantity units of one trichromatic system*, leaving the relative magnitudes of the various systems of units, where more than one system is involved, to be accounted for by the introduction of suitable coefficients in the purely arithmetical part of the work.

The problems dealt with are:

(*a*) Mixtures of two colours.

(*b*) Mixtures of three colours.

(*c*) Mixtures of four or more colours.

(*d*) Transformation of data from one trichromatic system to another, either directly or by means of general equations of transformation.

(*e*) Transformation of data between the trichromatic and monochromatic systems of specification.

(*f*) Calculation of colour from energy distribution of stimulus.

Graphical solutions are described for all these problems, non-graphical solutions being also given for all cases in which they are applicable. The graphical solutions are described in relation to rectangular co-ordinates so that they may be carried out on ordinary graph paper without the necessity of specially prepared charts.

The methods are illustrated by a number of worked examples specially designed to show how various types of data which may enter into a practical problem may be reduced to a form suitable for solution, and how the result of the colour calculation can be re-converted to the form in which the answer is required.

Introductory

In a recent paper* the author discussed algebraic transformations for the conversion of colour specifications from one trichromatic system to another. The purpose of the present paper is to show how all problems of colour mixture, including the transformations just referred to, may be solved by a system of calculation, based on plane geometry, which lends itself to graphical solution if desired.

* "The Transformation of Trichromatic Mixture Data: Algebraic Methods." *Trans. Opt. Soc.* 26 (1924–25) 95.

As pointed out in the previous paper, any colour whatever can be represented by an equation of the form

$$C = \alpha R + \alpha' G + \alpha'' B \qquad \ldots.(1),$$

where α, α', and α'', any of which may have negative values, are amounts of three "Primaries," which may be actual colours or hypothetical entities not separately perceivable. The only condition which the three primaries of a trichromatic system must fulfil, in order that the usual convention may be adopted in regard to the units in which α, α', and α'' are expressed, namely that equal quantities of the primaries make white, is that of being miscible to produce the sensation of white.

So long as the relative magnitudes of α, α', and α'' remain the same, variations of their actual magnitudes affect the luminosity but not the quality of the colour*. In dealing with the trichromatic specification of colours it is usual to define the *quantity* of a colour as the sum of the coefficients $\alpha + \alpha' + \alpha''$. Quantities, as thus defined, are not proportional to luminosity, except for colours of the same *quality*. In other words, the units of quantity for colours of different quality are not in general of equal luminosity.

We shall denote units of any colour, chosen in accordance with the above definition, as Trichromatic Units, or T. units, to distinguish them from units on a luminosity basis.

If in equation (1) the magnitudes of α, α', and α'' are so chosen that their sum is unity, the equation represents one T. unit of the colour C, and may be referred to as a unit equation, to distinguish it from a general trichromatic expression in which the coefficients may have any values.

Trichromatic quantities of different colours are additive. Let the unit equations of two colours be

$$C_1 = \alpha R + \alpha' G + \alpha'' B,$$
$$C_2 = \beta R + \beta' G + \beta'' B.$$

If m units of C_1 be mixed with n units of C_2 to produce p units of some third colour C_3, the total amounts of the primaries in the mixture are

$$(m\alpha + n\beta)\, R + (m\alpha' + n\beta')\, G + (m\alpha'' + n\beta'')\, B,$$

which, by hypothesis, represents pC_3.

The total quantity of the mixture colour is

$$\begin{aligned}&(m\alpha + n\beta) + (m\alpha' + n\beta') + (m\alpha'' + n\beta'')\\ &= m(\alpha + \alpha' + \alpha'') + n(\beta + \beta' + \beta'')\\ &= (m + n) \text{ T. units},\end{aligned}$$

whence $p = m + n$. Thus one T. unit of the mixture colour is given by

$$C = \left(\frac{m\alpha + n\beta}{m+n}\right) R + \left(\frac{m\alpha' + n\beta'}{m+n}\right) G + \left(\frac{m\alpha'' + n\beta''}{m+n}\right) B \qquad \ldots.(2).$$

* This statement, and the whole of the treatment which follows, supposes attention to be confined to the range of luminosity (of the final sensation) within which the colour vision of the eye is sensibly constant. The region of the "Purkinje Effect" (dark adaptation) and the region of dazzle are both excluded, as is usual in dealing with colorimetric problems.

The Colour Triangle

An equilateral triangle has the property that the sum of the perpendicular distances from the three sides of the triangle to any point in its plane is constant, distances measured outwards being regarded as negative. The constant sum is equal to the perpendicular from any corner to the opposite side. If then we take this perpendicular as our unit of length, we can represent colours by points in the plane of such a triangle if the distance from one side represents the coefficient of R in the unit equation of the colour, and the distances from the other two sides represent the coefficients of G and B respectively. The primaries themselves are obviously located at the three corners of the triangle, while white, which by the usual choice of units contains equal quantities of the primaries, is located at the centroid of the triangle.

Since the sum of the three co-ordinates in such a diagram is constant (equal to unity), all points in it represent one T. unit of the corresponding colours.

The device of plotting colours in such a diagram originated with Newton, and was employed by Maxwell, Helmholtz, Koenig, and others. The properties of the colour triangle, and of the three-dimensional colour pyramid, of which the colour triangle is a basal section, are fully developed in a paper by H. E. Ives*. In a later paper† the same author describes how he utilizes these for the graphical solution of a number of colorimetric problems.

The present writer employs a procedure which differs from that of Ives and is in some respects simpler in its application to practical problems. As geometrical methods are in many cases preferable to purely algebraic‡ methods it may be useful to give an account of a system of colour calculation by means of which every practical problem of colour mixture may be reduced to a problem in plane geometry. In every case, except the determination of hue, it is possible to calculate the results without recourse to graphical methods. As the balance of advantage, in particular circumstances, may lie with either, both graphical and non-graphical methods are indicated for each problem considered.

Rectangular Co-ordinates

While the colour triangle is the natural framework in which to plot colour, inasmuch as the position of a point on the diagram conveys at a glance, by its relative remoteness from the three sides of the triangle, the constitution of the colour in terms of the trichromatic primaries for which the diagram is constructed, it is an inconvenient reference frame from the point of view of ease and accuracy in plotting points or in reading their positions.

Moreover, regarded not as a section of the colour pyramid but merely as a

* *Journ. Franklin Inst.* 180 (Dec. 1915) 673. † *Ibid.* 195 (Jan. 1923) 23.

‡ The distinction here drawn, for convenience, between geometry and algebra is the usual one that in the latter the various variables may in general have any values, whereas in the former they are restricted by the condition that the points represented by the expressions also fulfil some geometrical relationship, such, for instance, as lying on a specified surface. Such restrictions of the generality of expressions usually lead to appreciable simplification of calculations.

framework in which to plot colours, the triangle provides a superfluous co-ordinate, since a point can be located in the diagram by any two of its trilinear co-ordinates. The restriction imposed by the fact that we are dealing with unit quantities of colour, for which the sum of the trichromatic coefficients is unity, has reduced the number of independent variables from three to two, and a colour is uniquely defined by any two of the coefficients in its unit equation.

This being so, we can plot colours in a chart with ordinary rectangular axes, taking any two of the coefficients in the unit equation as co-ordinates. Which pair we choose is theoretically immaterial; but, as we shall see later, there may in some cases be practical advantages in taking one pair rather than another for the solution of a particular problem. In the diagrams of the present paper the R and G coefficients are taken as co-ordinates, but this does not imply any general superiority of this arrangement over the other two.

Since, in general, trichromatic equations may contain one or more negative coefficients, colours may lie in any quadrant of the diagram. The two primaries which are taken as independent variables are themselves located on their respective axes at unity on the co-ordinate scales. The third primary is obviously located at the point corresponding to zero values for the other two, that is, at the origin of co-ordinates.

It may be as well to point out at this stage that the use of rectangular co-ordinates instead of the colour triangle introduces no essential modification in the methods to be followed. The relationships to be utilized hold whichever method of plotting is adopted, and all the essential processes can be carried out on the colour triangle in exactly the same way as on the other type of diagram.

The advantages of rectangular axes are purely those of practical convenience. Ordinary squared paper can be used, whereas in working with the colour triangle at least two of the scales must be specially constructed.

Further, although it is often convenient to employ the same scales for ordinates and abscissae, it is not essential to do this, and in special cases the accuracy of a graphical calculation may be increased by employing different scales in the two directions. There is no corresponding method of varying the arrangement of points in trilinear co-ordinates.

The first published reference to the use of rectangular co-ordinates for plotting the phenomena of colour occurs in the Report of the Colorimetry Committee of the Optical Society of America*, in which the colour triangle is represented in these co-ordinates. The third (oblique) side of the triangle has largely lost its significance in such a plot. It is not used as a reference line, and serves no purpose in the diagram beyond indicating the boundary of the area within which colours may actually be matched by positive quantities of the primaries (see later). Usually it need not be drawn, and the term "colour triangle" may suitably be replaced by the more general term "colour chart" when referring to a two-co-ordinate plot in which the third co-ordinate does not appear except as an implied function of the two others.

* *Journ. Opt. Soc. Amer. and Rev. Scient. Inst.* 6 (August, 1922) 527.

Mixture of Two Colours

The fundamental problem, to which every other problem of colour mixture is ultimately reducible, is to find the resultant colour produced by mixing two constituent colours in given proportions.

We have seen, equation (2), the constitution of one T. unit of the colour produced by mixing two other colours in the proportion m T. units of one to n T. units of the other.

Let C_1 and C_2 (Fig. 1) be the constituent colours plotted in R, G co-ordinates. If C_3 is a point which divides the straight line joining C_1C_2 in the ratio $C_3C_2/C_1C_3 = m/n$, its co-ordinates, γ and γ', are given, by a well-known geometrical theorem, as

$$\gamma = \frac{m\alpha + n\beta}{m+n}, \quad \gamma' = \frac{m\alpha' + n\beta'}{m+n}.$$

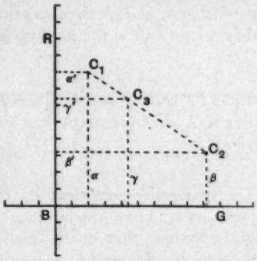

Fig. 1

But these are the co-ordinates of the colour represented by equation (2), so that C_3 represents the colour produced by mixing C_1 and C_2 in the ratio C_3C_2/C_1C_3 (in T. units). The foregoing result holds, of course, whatever the signs of the various coefficients, that is to say, in whichever quadrant of the chart any or all of the colours may lie, provided all quantities are given their proper signs.

It is obvious that all colours which can be produced by mixing C_1 and C_2 lie between C_1 and C_2 on the straight line joining them.

The same mathematical relationship between the coefficients holds when the ratio m/n is negative, that is, when C_3 lies on C_1C_2 produced in either direction. Such a relationship does not represent a possible colour mixture, since we cannot add a negative quantity of a colour, but it specifies the third colour in terms of the other two, which, as we shall see later, is sometimes useful.

Stated generally, the result we have just obtained may be put in the following form. Any one of three collinear colours, such as C_1, C_3, and C_2 of Fig. 1, may be *expressed* in terms of the other two, as follows:

$$C_1 = -\left(\frac{C_1C_3}{C_3C_2}\right)C_2 + \left(\frac{C_1C_2}{C_3C_2}\right)C_3 \qquad \ldots.(3),$$

$$C_3 = \left(\frac{C_3C_2}{C_1C_2}\right)C_1 + \left(\frac{C_1C_3}{C_1C_2}\right)C_2 \qquad \ldots.(4),$$

$$C_2 = \left(\frac{C_1C_2}{C_1C_3}\right)C_3 - \left(\frac{C_3C_2}{C_1C_3}\right)C_1 \qquad \ldots.(5).$$

The quantities in these equations are, of course, measured in the trichromatic quantity units, as previously defined, pertaining to the particular trichromatic system to which the chart refers. For many problems it is never necessary to go outside such a system of units, but we are not confined in our calculations to these,

since conversion to or from any other system of units can be accomplished by the introduction of numerical factors.

For instance, suppose we wish to express results obtained from the colour chart in terms of *luminosity*, that is, in units which represent equally bright quantities of the various colours concerned. If two colours C_1 and C_2 are mixed in the proportion m/n, as expressed in trichromatic units, the relative luminosity in which they are present in the mixture is

$$\frac{m}{n} \times \frac{L_{C_1}}{L_{C_2}},$$

where L_{C_1} and L_{C_2} are quantities proportional to the luminosities of one trichromatic unit of C_1 and C_2 respectively. Further, the luminosity of one T. unit of the mixture colour is

$$\frac{mL_{C_1} + nL_{C_2}}{m+n}.$$

The luminosity coefficient L_C of a T. unit of any colour C is readily obtained provided we know the luminosity coefficients of the units of the three primaries of the system, that is, the relative luminosities in which they match white. Let these be L_R, L_G, and L_B respectively; then, if

$$C = \alpha R + \alpha' G + \alpha'' B,$$

the luminosity of C, which is the sum of the luminosities of its constituents, is

$$L_C = \alpha L_R + \alpha' L_G + \alpha'' L_B.$$

As we are concerned with relative values only, it is customary to reduce values of L_R, L_G, and L_B so that their sum is unity.

With three such coefficients known for the particular trichromatic system in use, data for any colour are immediately convertible from trichromatic quantity units to luminosity units, or vice versa. This may best be illustrated by a concrete example.

Suppose we have two colours, known by their quality equations on a certain trichromatic system as

$$C_1 = \cdot 60R \quad + \cdot 15G + \cdot 25B,$$
$$C_2 = -\cdot 10R + \cdot 75G + \cdot 35B.$$

Assuming the luminosity coefficients of the primaries to be $L_R = \cdot 28, L_G = \cdot 70, L_B = \cdot 02$, what is the quality of the colour produced by mixing a quantity of C_1 with a quantity of C_2 which is only half as bright?

We must first find the required proportions in trichromatic units.

$$\begin{aligned} L_{C_1} &= \cdot 60L_R + \cdot 15L_G + \cdot 25L_B \\ &= \cdot 168 \quad + \cdot 105 \quad + \cdot 005 \\ &= \cdot 278. \end{aligned}$$

$$\begin{aligned} L_{C_2} &= -\cdot 10L_R + \cdot 75L_G + \cdot 35L_B \\ &= -\cdot 028 \quad + \cdot 525 \quad + \cdot 007 \\ &= \cdot 504. \end{aligned}$$

$$\begin{aligned} \frac{C_1}{C_2} &= 2, \text{ in luminosity units} \\ &= 2 \times \frac{L_{C_2}}{L_{C_1}} \text{ in T. units} \\ &= 3 \cdot 63 \text{ in T. units.} \end{aligned}$$

From equation (2) the coefficient of R in the mixture colour is

$$\frac{3{\cdot}63 \times {\cdot}60 - 1 \times {\cdot}10}{4{\cdot}63} = {\cdot}449,$$

that of G is

$$\frac{3{\cdot}63 \times {\cdot}15 + 1 \times {\cdot}75}{4{\cdot}63} = {\cdot}280,$$

that of B is

$$\frac{3{\cdot}63 \times {\cdot}25 + 1 \times {\cdot}35}{4{\cdot}63} = {\cdot}271.$$

It is only necessary to evaluate two of these; the third is obtainable by subtracting their sum from unity. The mixture colour is therefore

$$C_3 = {\cdot}449R + {\cdot}280G + {\cdot}271B.$$

The problem may be solved graphically (Fig. 2) by plotting C_1 and C_2 and marking off C_2C_3 equal to $C_1C_2 \times \frac{3{\cdot}63}{4{\cdot}63}$. C_3 is the required mixture colour, and its R and G coefficients are read directly from the chart. Its B coefficient is unity minus the sum of these.

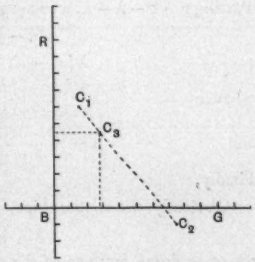

Fig. 2

It is obvious that in a simple problem such as the above, involving only a two-constituent mixture, there is nothing to be gained by actually plotting the colours in a diagram and employing the graphical method, as the arithmetical calculation only takes a few seconds. The example serves, however, to show the fundamental relationship between the facts of colour mixture and the geometry of the colour chart.

Mixture of Three Colours

Suppose we have three colours C_1, C_2, and C_3, known by their unit equations on a trichromatic system, and that we wish to find the colour C_4 produced by mixing these in given proportions. Let λ, μ, and ν be quantities proportional to the actual amounts of C_1, C_2, and C_3 in the mixture, but chosen so that $\lambda + \mu + \nu$ is unity, for simplification of the expressions. Let

$$C_1 = \alpha R + \alpha' G + \alpha'' B,$$
$$C_2 = \beta R + \beta' G + \beta'' B,$$
$$C_3 = \gamma R + \gamma' G + \gamma'' B,$$

and

$$C_4 = \delta R + \delta' G + \delta'' B.$$

Our problem is to determine δ, δ', and δ''.

By hypothesis

$$\begin{aligned} C_4 &= \lambda C_1 + \mu C_2 + \nu C_3 \\ &= (\lambda\alpha + \mu\beta + \nu\gamma)\, R \\ &\quad + (\lambda\alpha' + \mu\beta' + \nu\gamma')\, G \\ &\quad + (\lambda\alpha'' + \mu\beta'' + \nu\gamma'')\, B, \end{aligned}$$

whence
$$\left.\begin{aligned}\delta &= \lambda\alpha + \mu\beta + \nu\gamma\\ \delta' &= \lambda\alpha' + \mu\beta' + \nu\gamma'\\ \delta'' &= \lambda\alpha'' + \mu\beta'' + \nu\gamma''\end{aligned}\right\} \quad \ldots.(6).$$

The converse problem, that of determining the proportions of C_1, C_2, and C_3 which express another known colour C_4, involves the determination of λ, μ, and ν from the known equations of the four colours.

Since $\lambda + \mu + \nu = 1$, only two of them may be treated as independent variables. Putting $\nu = 1 - \lambda - \mu$, we may write the first two, say, of equations (6) as

$$\lambda(\alpha - \gamma) + \mu(\beta - \gamma) + \gamma - \delta = 0,$$
$$\lambda(\alpha' - \gamma') + \mu(\beta' - \gamma') + \gamma' - \delta' = 0,$$

whence
$$\lambda = \frac{(\beta - \gamma)(\gamma' - \delta') - (\beta' - \gamma')(\gamma - \delta)}{(\alpha - \gamma)(\beta' - \gamma') - (\alpha' - \gamma')(\beta - \gamma)}$$

and
$$\mu = \frac{(\gamma - \delta)(\alpha' - \gamma') - (\gamma' - \delta')(\alpha - \gamma)}{(\alpha - \gamma)(\beta' - \gamma') - (\alpha' - \gamma')(\beta - \gamma)} \quad \ldots.(7)$$

Finally,
$$\nu = 1 - \lambda - \mu.$$

Obviously, the relationship
$$\lambda C_1 + \mu C_2 + \nu C_3 - C_4 = 0 \quad \ldots.(8)$$
serves to define any one of the four colours in terms of the other three. If the evaluated coefficients of any three which express a positive quantity of the fourth are all positive, then the fourth can actually be matched by a mixture of the three, but if any are negative such a match is impossible.

The foregoing problems lend themselves to easy graphical solution. Plot the colours C_1, C_2, and C_3 on the chart (Fig. 3). To find C_4, the colour due to mixing

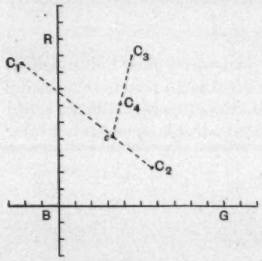

Fig. 3

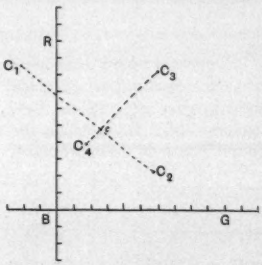

Fig. 4

these in the proportions $\lambda:\mu:\nu$, we proceed in two steps, applying the law of two-constituent mixture for each step.

Join C_1C_2 and mark off the point c, such that $cC_2 = C_1C_2 \times \frac{\lambda}{\lambda + \mu}$. The point c represents the colour produced by mixing λC_1 with μC_2. Join C_3c and mark off

C_4 so that $cC_4 = cC_3 \times \dfrac{\nu}{\lambda + \mu + \nu}$. C_4 represents the colour produced by mixing ν units of C_3 with $\lambda + \mu$ units of c, that is, with $\lambda C_1 + \mu C_2$, and is therefore the required mixture colour.

The converse problem of determining the proportions of C_1, C_2, and C_3 which will match a known colour C_4 is solved as follows. Join C_3 to C_4 and produce to meet the line joining C_1C_2 in c. C_4 may be regarded as a mixture of C_3 and the colour represented by c.

$$C_4 = \left(\frac{cC_4}{cC_3}\right) C_3 + \left(\frac{C_4C_3}{cC_3}\right) c.$$

But
$$c = \left(\frac{cC_2}{C_1C_2}\right) C_1 + \left(\frac{C_1c}{C_1C_2}\right) C_2,$$

whence
$$C_4 = \left(\frac{cC_4}{cC_3}\right) C_3 + \left(\frac{C_4C_3}{cC_3}\right) \left\{\left(\frac{cC_2}{C_1C_2}\right) C_1 + \left(\frac{C_1c}{C_1C_2}\right) C_2\right\}.$$

For instance, in the case depicted in Fig. 3,

$$\frac{cC_4}{cC_3} = \cdot 40,$$

whence
$$C_4 = \cdot 40C_3 + \cdot 60c.$$

Also
$$\frac{cC_2}{C_1C_2} = \cdot 30,$$

whence
$$c = \cdot 30C_1 + \cdot 70C_2.$$

Thus
$$C_4 = \cdot 40C_3 + \cdot 60\,(\cdot 30C_1 + \cdot 70C_2)$$
$$= \cdot 18C_1 + \cdot 42C_2 + \cdot 40C_3.$$

It is clear from the construction of Fig. 3 that C_4 can be matched by mixing C_1, C_2, and C_3, provided it lies anywhere within the triangle $C_1C_2C_3$. If it lies outside this triangle it cannot be matched by any possible mixture of these. This corresponds to a negative value for λ, μ, or ν, or possibly of two of them, in equation (8).

In such a case we can still determine graphically the *expression* for C_4 in terms of C_1, C_2, and C_3.

Join C_3C_4 and C_1C_2 (Fig. 4). Express C_4 in terms of C_3 and c, the colour at the point of intersection:

$$C_4 = \left(\frac{C_4C_3}{cC_3}\right) c - \left(\frac{C_4c}{cC_3}\right) C_3.$$

Also
$$c = \left(\frac{cC_2}{C_1C_2}\right) C_1 + \left(\frac{C_1c}{C_1C_2}\right) C_2,$$

from which the coefficients in the expression for C_4 are obtained. For example, in Fig. 4,

$$\frac{C_4C_3}{cC_3} = 1 \cdot 25,$$

whence
$$C_4 = 1 \cdot 25c - \cdot 25C_3.$$

Also $$\frac{cC_2}{C_1C_4} = \cdot 4,$$

whence $$c = \cdot 4C_1 + \cdot 6C_2$$

and $$C_4 = 1\cdot 25\,(\cdot 4C_1 + \cdot 6C_2) - \cdot 25C_3$$
$$= \cdot 50C_1 + \cdot 75C_2 - \cdot 25C_3.$$

Had the arrangement been such that C_3C_4 cut C_1C_2 externally, we should have found two of the coefficients negative, but the method is precisely the same in all cases.

We may note in passing that with any four given colours it is possible to arrange one, *and only one*, colour match.

Suppose the colours lie as in Fig. 3, in which case one colour is within the triangle formed by three others. As we have seen, we can match C_4 by a mixture of C_1, C_2, and C_3. But we cannot match C_1 by a mixture of C_2, C_3, and C_4, since it lies outside the triangle formed by these. Similarly we cannot match C_2 or C_3 by combining the remainder. Thus there is only one match possible in which one of the colours is matched by the three others.

If the colours lie as in Fig. 4, in which none of the colours is within the triangle formed by the others, no one of them can be matched by a combination of the others, but if we illuminate one side of a photometer with a combination of C_1 and C_2, and the other side with a combination of C_3 and C_4, each pair being in the proportions which give the colour c, a colour match will be produced. These are the only pairs which have a common mixture colour, so that only one match is possible. Reverting to Fig. 3, we note that, in the arrangement depicted there, no match can be made by combining the colours in pairs. Hence, whatever the nature of the colours, one colour match, and only one, may be arranged with them. It may either be a match of one against three or of two against two, but both types of match cannot be arranged from the same four colours.

The results, which are obvious from the pictorial representation, can also be deduced from equation (8), because it is clear that, whatever the values and signs of λ, μ, and ν may be, there is only one way of arranging the four elements with respect to the equality sign so that all coefficients may be positive.

Thus in the case of Fig. 4, for which we have already found

$$C_4 = \cdot 50C_1 + \cdot 75C_2 - \cdot 25C_3,$$

the only colour match possible is the one expressed by

$$C_4 + \cdot 25C_3 = \cdot 50C_1 + \cdot 75C_2.$$

That is to say, 4 T. units of $C_4 + 1$ T. unit of C_3 will match 2 T. units of $C_1 + 3$ T. units of C_2.

To illustrate the use of the above relationships in practical problems, let us consider the following example:

Three lamps, of 15, 25, *and* 30 *candle-power respectively, are enclosed in coloured globes which have transmission factors of* 35, 50, *and* 5 *per cent. as measured, for incident light of the quality emitted by the lamps, with a flicker photometer.*

Measurements of the enclosed lamps on a particular trichromatic colorimeter gave the unit equations

$$C_1 = \cdot 6R + \cdot 2G + \cdot 2B,$$
$$C_2 = -\cdot 1R + \cdot 8G + \cdot 3B,$$
$$C_3 = \cdot 2R + \cdot 1G + \cdot 7B.$$

At what distances from a screen must these enclosed lamps be situated in order that the resulting illumination shall be white, and shall have an intensity of 5 metre-candles?

Take the luminosity coefficients of the colorimeter primaries as $L_R = \cdot 28$, $L_G = \cdot 70$, *and* $L_B = \cdot 02$.

The colour mixture part of this problem consists in finding the proportions in which three colours, given by these trichromatic equations, must be mixed to match white.

Let $$W = \lambda C_1 + \mu C_2 + \nu C_3.$$

Also $$W = \cdot\dot{3}R + \cdot\dot{3}G + \cdot\dot{3}B.$$

Making the required numerical substitutions in equation (7), we find that

$$\lambda = \cdot 527, \quad \mu = \cdot 258.$$

Therefore $$W = \cdot 527C_1 + \cdot 258C_2 + \cdot 215C_3$$

in the quantity units of the trichromatic system.

To convert these proportions to the luminosity basis we must, as we have seen, multiply each coefficient by the luminosity of one T. unit of each colour.

$$L_{C_1} = \cdot 6L_R + \cdot 2L_G + \cdot 2L_B = \cdot 312,$$
$$L_{C_2} = \cdot 1L_R + \cdot 8L_G + \cdot 3L_B = \cdot 458,$$
$$L_{C_3} = \cdot 2L_R + \cdot 1L_G + \cdot 7L_B = \cdot 140,$$

whence the relative luminosities of the constituents required to match white are

$$\cdot 527L_{C_1} : \cdot 258L_{C_2} : \cdot 215L_{C_3} = \cdot 164 : \cdot 118 : \cdot 030.$$

The total illumination required is 5 metre-candles, to which each lamp must contribute in the above proportions. The illuminations from the individual lamps are therefore 2·63, 1·89, and 0·48 metre-candles respectively.

The effective candle-powers of the enclosed lamps are

$$15 \times \cdot 35 = 5 \cdot 25, \quad 25 \times \cdot 50 = 12 \cdot 5, \quad \text{and} \quad 30 \times \cdot 05 = 1 \cdot 5$$

respectively. The distances from a screen at which these lamps will produce the required illuminations are, therefore,

$$\sqrt{\frac{5 \cdot 25}{2 \cdot 63}} = 1 \cdot 41, \quad \sqrt{\frac{12 \cdot 5}{1 \cdot 89}} = 2 \cdot 57, \quad \text{and} \quad \sqrt{\frac{1 \cdot 5}{\cdot 48}} = 1 \cdot 77 \text{ metres.}$$

The colour mixture part of the foregoing problem may be solved graphically

as follows. Plot C_1, C_2, and C_3 on a colour chart (Fig. 5). Join C_2C_3 and C_1W, producing the latter to meet C_2C_3 at c.

The proportion of C_1 in the mixture which makes white is $cW/cC_1 = \lambda$, say. The remaining $1 - \lambda$ contains C_2 in the proportion cC_3/C_2C_3, so that

$$\mu = (1 - \lambda)\frac{cC_3}{C_2C_3}.$$

The following is another example on somewhat different lines.

There are four lamps, of which the colour qualities and candle-powers are

$$C_1 = \cdot 70R + \cdot 20G + \cdot 10B; \text{ candle-power} = 2\cdot 5,$$
$$C_2 = -\cdot 20R + \cdot 85G + \cdot 35B; \quad \text{,,} \quad = 1\cdot 5,$$
$$C_3 = \cdot 15R + \cdot 80G + \cdot 05B; \quad \text{,,} \quad = 2\cdot 0,$$
$$C_4 = \cdot 60R + \cdot 10G + \cdot 30B; \quad \text{,,} \quad = 3\cdot 0.$$

How must these lamps be arranged with respect to a photometer head in order to secure a match both in colour and brightness with a screen illumination of 10 metre-candles?

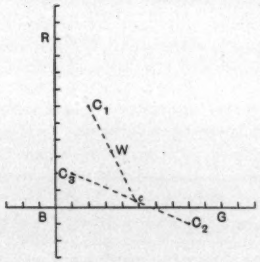

Fig. 5

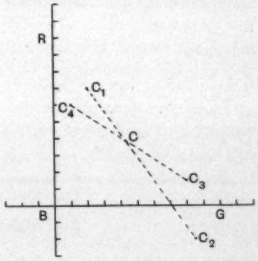

Fig. 6

Plot the four colours (Fig. 6). It is at once evident that the colour match which can be arranged with these lamps is that in which C_1 and C_2 are on one side of the photometer, while C_3 and C_4 are on the other. The colour of the match will be C, and the required proportions, in trichromatic units, are

$$C_1/C_2 = \frac{CC_2}{C_1C} = 1\cdot 935 \text{ and } C_3/C_4 = \frac{C_4C}{CC_3} = \cdot 849.$$

Taking the same values for the luminosity factors of the trichromatic primaries as in the previous problem, we find that

$$L_{C_1} = \cdot 338,\ L_{C_2} = \cdot 546,\ L_{C_3} = \cdot 603, \text{ and } L_{C_4} = \cdot 244.$$

The relative luminosities of the constituents in the mixtures are therefore

$$C_1/C_2 = 1\cdot 935 \times \frac{\cdot 338}{\cdot 546} = 1\cdot 195, \quad C_3/C_4 = \cdot 849 \times \frac{\cdot 603}{\cdot 244} = 2\cdot 095.$$

The illumination of the photometer screen due to C_1 and C_2 has to be 10 metre-candles. The illumination due to C_1 is therefore $10 \times \frac{1{\cdot}195}{2{\cdot}195} = 5{\cdot}45$, that due to C_2 being $4{\cdot}55$. The illumination on the other side of the photometer is also 10 metre-candles, of which C_3 contributes $10 \times \frac{2{\cdot}095}{3{\cdot}095} = 6{\cdot}77$ and C_4 contributes $3{\cdot}23$. The distances at which the lamps must be placed are, therefore,

$$\sqrt{\frac{2{\cdot}25}{5{\cdot}45}} = {\cdot}676, \quad \sqrt{\frac{1{\cdot}5}{4{\cdot}55}} = {\cdot}574, \quad \sqrt{\frac{2{\cdot}0}{6{\cdot}77}} = {\cdot}544, \text{ and } \sqrt{\frac{3{\cdot}0}{3{\cdot}23}} = {\cdot}963 \text{ metre.}$$

We could have solved the colour mixture part of the problem non-graphically by expressing one of the colours, say C_4, in terms of the other three. Applying equations (7) to the data we find

$$C_4 = 1{\cdot}219C_1 + {\cdot}630C_2 - {\cdot}849C_3$$

or, arranging so as to avoid negative coefficients,

$$C_4 + {\cdot}849C_3 = 1{\cdot}219C_1 + {\cdot}630C_2,$$

which represents the only colour match possible with these four colours.

The ratio of C_1 to C_2 in the one mixture is $1{\cdot}219/{\cdot}630 = 1{\cdot}935$; the ratio of C_3 to C_4 in the other is ${\cdot}849$.

The foregoing examples serve to show how actual problems are reduced to a form suitable for the application of the laws of colour mixture. We see that all data of an extraneous character may be eliminated by mere arithmetic, reducing the problem to those elements which are amenable to treatment by plane geometry.

Mixtures of Four or More Colours

If we have four colours

$$\begin{aligned} C_1 &= \alpha R + \alpha' G + \alpha'' B, \\ C_2 &= \beta R + \beta' G + \beta'' B, \\ C_3 &= \gamma R + \gamma' G + \gamma'' B, \\ C_4 &= \delta R + \delta' G + \delta'' B, \end{aligned}$$

and it is desired to mix these in the proportions $\lambda C_1 + \mu C_2 + \nu C_3 + \omega C_4$, the resulting colour is clearly

$$\begin{aligned} C_5 = &\left(\frac{\lambda\alpha + \mu\beta + \nu\gamma + \omega\delta}{\lambda + \mu + \nu + \omega}\right) R \\ &+ \left(\frac{\lambda\alpha' + \mu\beta' + \nu\gamma' + \omega\delta'}{\lambda + \mu + \nu + \omega}\right) G \\ &+ \left(\frac{\lambda\alpha'' + \mu\beta'' + \nu\gamma'' + \omega\delta''}{\lambda + \mu + \nu + \omega}\right) B \qquad \ldots\ldots(9). \end{aligned}$$

As in the case of trichromatic mixtures, the graphical method of determining the mixture colour consists of successive applications of the procedure for two colours. The colours are plotted and joined in separate pairs, say C_1C_2 and C_3C_4.

The point c (Fig. 7) which divides C_1C_2 so that $C_2c/cC_1 = \lambda/\mu$ and the point c' which divides C_3C_4 so that $C_4c'/c'C_3 = \nu/\omega$ are found. The point C_5 which divides cc' so that $c'C_5/C_5c = (\lambda + \mu)/(\nu + \omega)$ is the colour due to mixing all four.

The converse problem of determining in what proportions four colours must be mixed to match a fifth has clearly an infinite number of solutions. We can find one solution by drawing through C_5 any straight line which cuts both C_1C_2 and C_3C_4 internally. Let the points of intersection be c and c' (Fig. 7). Then, if λ, μ, ν, and ω are quantities such that

$$\lambda/\mu = cC_2/C_1c,\ \nu/\omega = c'C_4/C_3c',\text{ and } (\lambda + \mu)/(\nu + \omega) = C_5c'/cC_5,$$

$$(\lambda + \mu + \nu + \omega)\, C_5 = \lambda C_1 + \mu C_2 + \nu C_3 + \omega C_4.$$

The above procedure is always possible provided C_5 does not lie outside the quadrilateral $C_1C_2C_3C_4$.

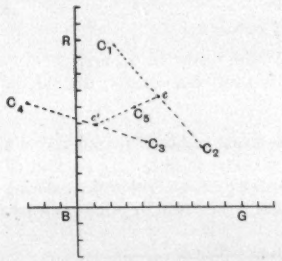

Fig. 7

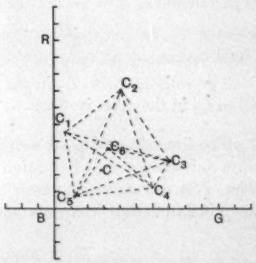

Fig. 8

In general the colours producible by mixing any number of constituent colours lie within the smallest polygon (without re-entrant angles) which contains all the constituents. Conversely any colour within this polygon may be matched by mixing the constituent colours in suitable proportions. If there are more than three constituents the match may be effected in an indefinite number of ways. Further, there is always one or more of these ways in which all but three of the constituents have zero values, because any point within the smallest polygon containing a number of points must lie within a triangle formed by some three of them, and may consequently be matched by these three alone. For instance the colour C (Fig. 8) may be matched by various mixtures of the six colours C_1, C_2, C_3, C_4, C_5, and C_6. We can determine as many such mixtures as we choose by extending the method just described for mixtures of four colours. It is clear, however, that C can be matched by trichromatic mixtures of C_1, C_4, C_5; C_1, C_3, C_5; C_2, C_4, C_5; C_2, C_3, C_5; C_4, C_5, C_6; or C_3, C_5, and C_6. Any one of these groups of three can match C without the aid of the remaining constituents. It is therefore never necessary to use more than three at one time from any group of available constituents in order to match any colour which it is possible to match at all with such constituents.

Transformation of Data from one Trichromatic System to Another

In the foregoing paragraphs it has been shown how any colour mixture problem can be solved by mathematical methods provided the specification of the colours in terms of a set of trichromatic primaries is known.

The principles involved find very important application in the transference of colour specifications from one system of specification to another. We shall first consider the case in which both systems are trichromatic, but involve different sets of primaries. This problem was treated by purely algebraical methods in the author's previous paper*. The geometrical relationships of the present paper may also be applied to such a transformation.

Suppose we have two trichromatic systems of which the primaries are R_1, G_1, B_1 and R_2, G_2, B_2 respectively. In order to transform data from one system to the other it is necessary to know the quality equations of one set of primaries in terms of the other, that is, to know the positions on the colour chart of one system occupied by the primaries of the other system.

Let R_2, G_2, and B_2 be known in terms of R_1, G_1, and B_1 by the relations

$$\left.\begin{aligned} R_2 &= aR_1 + a'G_1 + a''B_1 \\ G_2 &= bR_1 + b'G_1 + b''B_1 \\ B_2 &= cR_1 + c'G_1 + c''B_1 \end{aligned}\right\} \qquad \ldots\ldots(10),$$

where $a + a' + a'' = 1$, $b + b' + b'' = 1$, and $c + c' + c'' = 1$.

The quantities of R_2, G_2, and B_2 represented by these equations are not the units of quantity in which they are measured in their own system, but are the units in which they are expressed in the other system. Let us refer to the quantity units of the $R_1G_1B_1$ system as T_1. units, and the quantity units of the $R_2G_2B_2$ system as T_2. units. Equations (10), therefore, represent one T_1. unit of R_2, G_2 and B_2.

So long as we measure R_2, G_2, *and* B_2 *in* T_1. *units* the problem of expressing, in terms of R_2, G_2, and B_2, any colour known by its position on the chart (i.e. known in the $R_1G_1B_1$ system), and the converse problem of determining the co-ordinates on the $R_1G_1B_1$ system of a colour known to be a given mixture of R_2, G_2, and B_2, are merely examples of the type of three-colour mixture problem already explained and illustrated.

In order to convert quantities of R_2, G_2, and B_2 to or from the units of their own system it is only necessary to associate with each of these primaries a numerical factor proportional to the amount, measured in T_1. units, which corresponds to a T_2. unit.

We can determine these factors by finding the proportions of R_2, G_2, and B_2, measured in T_1. units, which match white. This again is merely a particular case of the problem already treated.

Suppose we find that

$$W = pR_2 + qG_2 + rB_2, \text{ in } T_1. \text{ units.}$$

By definition $$W = \cdot\dot{3}R_2 + \cdot\dot{3}G_2 + \cdot\dot{3}B_2, \text{ in } T_2. \text{ units.}$$

* *Loc. cit.*

Thus, assuming the same quantity of white to be taken as unity on both systems,

$3p$T$_1$. units of R_2 = one T$_2$. unit of R_2.

Similarly $3q$T$_1$. units of G_2 = one T$_2$. unit of G_2

and $3r$T$_1$. units of B_2 = one T$_2$. unit of B_2.

By means of these factors the change of units involved in converting data from one system to the other can be effected by simple arithmetic, leaving the geometrical processes to be carried out entirely in terms of one of the systems, that is to say, entirely in the plane of a single colour chart.

If, as the result of calculations on the $R_1G_1B_1$ system, we find the expression for a colour C, in terms of T$_1$. units of R_2, G_2, and B_2, to be

$$C = lR_2 + l'G_2 + l''B_2 \qquad \ldots.(11),$$

we can write this

$$C = \left(\frac{l}{3p}\right) R_2 + \left(\frac{l'}{3q}\right) G_2 + \left(\frac{l''}{3r}\right) B_2 \qquad \ldots.(12),$$

in which R_2, G_2, and B_2 are now expressed in T$_2$. units.

Equation (12) is not, however, a unit equation on the $R_2G_2B_2$ system, because it represents the same quantity of C as equation (11), that is, a T$_1$. unit of C, which, in general, differs from a T$_2$. unit of C. In order to reduce equation (12) so that it represents a T$_2$. unit of C we must divide each of the coefficients of R_2, G_2, and B_2 by the sum of all three.

Conversely, if we have a colour initially expressed in the $R_2G_2B_2$ system as

$$C = mR_2 + m'G_2 + m''B_2 \qquad \ldots.(13),$$

we can change this to

$$C = 3pmR_2 + 3qm'G_2 + 3rm''B_2 \qquad \ldots.(14),$$

in which R_2, G_2, and B_2 are now expressed in T$_1$. units.

This again is not a unit equation, since it represents a T$_2$. unit instead of a T$_1$. unit of C, but it can be reduced by dividing each coefficient by the sum of the three. When the data are in this form, the application of equation (6), or the graphical construction of Fig. 3, gives the expression for the colour C in terms of R_1, G_1, and B_1.

Thus, transformation of data may be effected *in either direction* between two trichromatic systems, of which the primaries of one are known by their unit equations in terms of the other. By making the actual colour mixture calculations with all quantities expressed in the units of one system the transformations are reduced to problems in plane geometry, thereby obviating the complications of stereographic projection of one colour triangle on another, construction of unevenly divided scales, etc. which enter into the method described by Dr Ives.

A practical example of a transformation in each direction will serve to illustrate the above procedure.

The quality of a colour is known, in terms of one set of primaries, as

$$C = \cdot 25R_1 + \cdot 10G_1 + \cdot 65B_1.$$

What is its specification in terms of another set of primaries which are themselves defined, in quality, by the following equations in terms of the first set?

$$R = \cdot 90R_1 + \cdot 20G_1 - \cdot 10B_1 \ldots \qquad (a),$$

$$G = \cdot 05R_1 + \cdot 75G_1 + \cdot 20B_1 \ldots \qquad (b),$$

$$B = \cdot 15R_1 - \cdot 10G_1 + \cdot 95B_1 \ldots \qquad (c).$$

To make this transformation graphically, plot R_2, G_2, B_2, and C in the colour chart of the $R_1G_1B_1$ system (Fig. 9). Join one of the new primaries, say G_2, to C, and produce to meet R_2B_2 in c. C contains G_2 in the proportion $cC/cG_2 = \cdot 180_5$ and the remainder ($\cdot 819_5$) contains R_2 in the proportions $cB_2/R_2B_2 = \cdot 192_5$, whence

$$C = \cdot 157_5R_2 + \cdot 180_5G_2 + \cdot 662_0B_2 \quad \ldots . (d),$$

where all quantities are expressed in T_1. units.

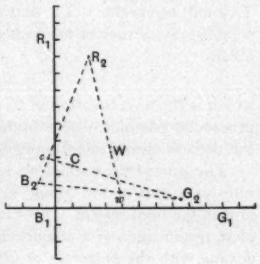

Fig. 9

To find the factors required for conversion of the units of R_2, G_2, and B_2 to those of their own system, join one of the new primaries, say R_2, to W and produce to meet G_2B_2 in w. Then W contains R_2 in the proportion $\frac{Ww}{R_2w} = \cdot 298$, and the remainder ($\cdot 702$) contains G_2 in the proportion $\frac{wB_2}{G_2B_2} = \cdot 576_5$, whence

$$W = \cdot 298_4R_2 + \cdot 404_6G_2 + \cdot 297_0B_2.$$

Thus

$\cdot 298_4$ T_1. units of $R_2 = \frac{1}{3}$ T_2. units

$\cdot 404_6$ T_1. „ $G_2 = \frac{1}{3}$ T_2. „

$\cdot 297_0$ T_1. „ $B_2 = \frac{1}{3}$ T_2. „

Thus equation (d) becomes, when R_2, G_2, and B_2 are expressed in T_2. units,

$$C = \left(\frac{\cdot 157_5}{3 \times \cdot 298_4}\right)R_2 + \left(\frac{\cdot 180_5}{3 \times \cdot 404_6}\right)G_2 + \left(\frac{\cdot 662_0}{3 \times \cdot 297_0}\right)B_2$$

$$= \cdot 175_9R_2 + \cdot 148_7G_2 + \cdot 743_0B_2 \qquad \ldots . (e).$$

This still represents a T_1. unit of C. By dividing the coefficients in equation (e) by their sum, namely 1·0676, we obtain, finally,

$$C = \cdot 165R_2 + \cdot 139G_2 + \cdot 696B_2 \qquad \ldots . (f),$$

in which all quantities are in the units of the $R_2G_2B_2$ system.

Consider, now, an example of a reverse transformation between the same two systems. Suppose we have a colour C' known by its unit equation on the $R_2G_2B_2$ system

$$C' = \cdot 6R_2 - \cdot 1G_2 + \cdot 5B_2 \qquad \ldots . (g).$$

We must change the units of R_2, G_2, and B_2 to those of the $R_1G_1B_1$ system before the data can be dealt with on the $R_1G_1B_1$ chart (or by the corresponding formulae).

We determine the ratios of the units as in the previous case; then (*g*) may be written

$$C' = (\cdot 6 \times 3 \times \cdot 298_4)\, R_2 - (\cdot 1 \times 3 \times \cdot 404_6)\, G_2 + (\cdot 5 \times 3 \times \cdot 297_0)\, B_2$$
$$= \cdot 537_1 R_2 - \cdot 121_4 G_2 + \cdot 445_5 B_2 \qquad \ldots\ldots(h),$$

in which R_2, G_2, and B_2 are in T_1. units. Multiplying the coefficients of R_1, G_1, and B_1 in equations (*a*), (*b*), and (*c*) by the appropriate coefficients in equation (*h*) and adding, we find

$$C' = \cdot 5442 R_1 - \cdot 0282 G_1 + \cdot 3452 B_1 \qquad \ldots\ldots(k).$$

This still represents a T_2. unit of C'. In order that it may represent a T_1. unit, we must divide each of the coefficients in equation (*k*) by their sum, ·8612, when we obtain

$$C' = \cdot 632 R_1 - \cdot 033 G_1 + \cdot 401 B_1 \qquad \ldots\ldots(l),$$

which is the unit equation of C' on the $R_1 G_1 B_1$ system. We could, of course, have proceeded graphically after obtaining the information in the form of equation (*h*), but there is clearly no advantage in doing so in such a case.

The sum of the coefficients in an equation such as (*e*) or (*k*) obviously gives the ratio of a unit of the colour in question in the initial trichromatic system to its unit in the final system. This ratio is of use if it is required to transform the complete specification of a colour, and not merely its quality. In cases where we are dealing with the *properties of illuminants*, it is usual to leave the brightness factor out of account in colorimetric specifications, such factor being attended to, when necessary, by other means, and expressed in separate terminology. Thus, in measuring the colour of the light given by a lamp, the actual settings of a colorimeter required to match this are immaterial; it is their *ratios*, expressed in proper units, in which we are interested, and the unit equation deduced from such readings tells us all we want to know about the colour.

It is different, however, when the colour specification refers to the *properties of a material* as, for example, a piece of coloured glass or coloured paper. In such cases it is necessary to take into consideration not only the quality of the transmitted or reflected light, but also the quantity of it in relation to the quantity of the incident illumination.

A trichromatic equation in which the sum of the coefficients represents the quantity (in trichromatic units) of a colour in relation to some arbitrary standard of quantity may be termed a total equation to distinguish it from a unit equation which indicates the quality only. Total equations can always be converted to unit equations by dividing throughout by the total quantity, but unit equations cannot be converted to total equations without external information as to the required quantity.

The conversion of total equations from one trichromatic system to another may be effected by purely arithmetical modification of the results obtained in terms of unit equations.

This again may best be illustrated by a practical example. Let us suppose that the colour *C* (Fig. 9) relates to a material, and that it has been measured on

a colorimeter of which R_1, G_1, and B_1 are the primaries. Such an instrument will be provided with scales of some kind on which the amount of each primary involved in a particular setting is indicated. The readings of these scales are not, of course, directly comparable; they have to be correlated by measurements on the standard white.

Let the scale readings when the instrument is set to match the standard white be

$$\left.\begin{array}{ll} R_1 \text{ reads} & 90{\cdot}0 \\ G_1 \quad ,, & 120{\cdot}0 \\ B_1 \quad ,, & 100{\cdot}0 \end{array}\right\} \left\{\begin{array}{l} \text{Divisions, or other measurement} \\ \text{units peculiar to the instrument} \end{array}\right.$$

and let the readings when the coloured material is substituted for the standard white be

$$\begin{array}{ll} R_1 \text{ reads} & 38{\cdot}0, \\ G_1 \quad ,, & 20{\cdot}25, \\ B_1 \quad ,, & 109{\cdot}5. \end{array}$$

Taking the quantity of the standard white as unity, we note that $\cdot\dot{3}$ unit of the respective primaries is represented by 90, 120, and 100 on the scales, so that the total equation of the colour is

$$\begin{aligned} mC &= \frac{38{\cdot}0}{270}R_1 + \frac{20{\cdot}25}{360}G_1 + \frac{109{\cdot}5}{300}B_1 \\ &= {\cdot}141R_1 + {\cdot}056G_1 + {\cdot}365B_1, \end{aligned}$$

whence the quantity m is ·562 T_1. units.

Deducing the unit equations from this we proceed to transform to the $R_2G_2B_2$ system, as we have already done in the first of these examples. We found the unit equation on the second system to be

$$C = {\cdot}165R_2 + {\cdot}139G_2 + {\cdot}696B_2.$$

To obtain the total equation we multiply these coefficients by the total quantity. This is ·562 in T_1. units of C; but we have seen that one T_1. unit of $C = 1{\cdot}068$ T_2. units of C, so that the quantity of the colour in the second system is

$$\cdot562 \times 1{\cdot}068 = \cdot600 \text{ unit.}$$

The total equation on this system is, therefore,

$$\cdot600C = \cdot099R_2 + \cdot083G_2 + \cdot418B_2.$$

Suppose now that a second colorimeter has R_2, G_2, and B_2 as its primaries, and that its scale readings when matching the standard white are

$$\begin{array}{ll} R_2 \text{ reads} & 40{\cdot}0, \\ G_2 \quad ,, & 35{\cdot}0, \\ B_2 \quad ,, & 65{\cdot}0. \end{array}$$

Reasoning as before, one unit of R_2 is represented by a scale reading of 120, one of G_2 by a reading of 105, and one of B_2 by a reading of 195. The actual settings of this instrument which would match the colour in question are therefore

$$\begin{array}{ll} R_2 \text{ reads} & 120 \times \cdot099 = 11{\cdot}9, \\ G_2 \quad ,, & 105 \times \cdot084 = 8{\cdot}8_2, \\ B_2 \quad ,, & 195 \times \cdot419 = 81{\cdot}7. \end{array}$$

The colour considered in the above transformation is one which can actually be matched on both instruments, but the same procedure enables us to compare, in quantity as well as in quality, the specifications of any colour whatever as obtained on two different trichromatic systems.

The method of transformation just described, in which the geometrical processes are applied directly to the colour in question, will probably be found to effect a considerable saving of time, as compared with the algebraic methods of the previous paper, in cases where only a few colours have to be transferred between two systems. On the other hand, where many transformations have to be made, the work involved in calculating the coefficients in equations of transformation such as are developed in the previous paper has only to be done once and serves for all transformations between the same two systems, the work of any individual transformation being simply numerical.

Incidentally, the coefficients in such equations of transformation may themselves be determined by the methods of the present paper. We shall illustrate this by deriving such equations for the systems dealt with in the previous examples.

Our initial information is

$$R_2 = \cdot 90R_1 + \cdot 20G_1 - \cdot 10B_1 \qquad \ldots . (a),$$
$$G_2 = \cdot 05R_1 + \cdot 75G_1 + \cdot 20B_1 \qquad \ldots . (b),$$
$$B_2 = \cdot 15R_1 - \cdot 10G_1 + \cdot 95B_1 \qquad \ldots . (c),$$

representing a T_1. unit of each of the No. 2 primaries. Determine the proportions, measured in T_1. units, in which R_2, G_2, and B_2 match white. We have already done this in the previous examples, and have found

$$\cdot 298_4\ T_1.\text{ units of } R_2 = \cdot\tfrac{3}{3}\ T_2.\text{ units}$$
$$\cdot 404_6 \quad ,, \quad G_2 = ,, \quad ,,$$
$$\cdot 297_0 \quad ,, \quad B_2 = ,, \quad ,,$$

Multiplying the coefficients in equation (*a*) by $3 \times \cdot 298_4$, those of equation (*b*) by $3 \times \cdot 404_6$ and of equation (*c*) by $3 \times \cdot 297_0$, we get

$$R_2 = \cdot 8057R_1 + \cdot 1791G_1 - \cdot 0895B_1 \qquad \ldots . (a_1),$$
$$G_2 = \cdot 0606R_1 + \cdot 9096G_1 + \cdot 2426B_1 \qquad \ldots . (b_1),$$
$$B_2 = \cdot 1337R_1 - \cdot 0891G_1 + \cdot 8465B_1 \qquad \ldots . (c_1),$$

where R_2, G_2, and B_2 are in T_2. units.

These equations, in which the primaries of each set are measured in the units of their own systems, are in the form required for transforming data from the $R_2G_2B_2$ system to the $R_1G_1B_1$ system by multiplication and collection of coefficients, as explained in the previous paper.

The departures from unity of the sums of the coefficients of R_1, G_1, and B_1, namely ·0000, ·0004 and ·0004, indicate the accuracy with which the various calculations leading up to equations (a_1), (b_1), and (c_1) have been carried out.

The equations for the reverse transformation are obtained by expressing R_1, G_1, and B_1 in terms of R_2, G_2, and B_2.

Join R_1 to one of the No. 2 primaries, say G_2 (Fig. 10). Let R_1G_2 intersect the line R_2B_2 joining the other No. 2 primaries at r. We first express R_1 in terms of G_2 and the colour at r.

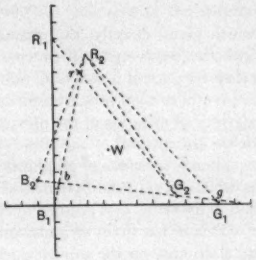

Fig. 10

$$rR_1/G_2r = -\cdot2697.$$

$$\therefore \quad R_1 = -\cdot2697G_2 + 1\cdot2697r.$$

Also $rR_2/B_2R_2 = \cdot1357.$

$$\therefore \quad r = \cdot1357B_2 + \cdot8643R_2,$$

$$1\cdot2697r = \cdot1723B_2 + 1\cdot0974R_2,$$

whence $\quad R_1 = 1\cdot0974R_2 - \cdot2697G_2 + \cdot1723B_2.$

Join G_1R_2, intersecting B_2G_2 in g.

$$G_1g/gR_2 = -\cdot0262.$$

$$\therefore \quad G_1 = -\cdot0262R_2 + 1\cdot0262g.$$

Also $gG_2/G_2B_2 = -\cdot2700.$

$$\therefore \quad g = -\cdot2700B_2 + 1\cdot2700G_2,$$

whence $\quad G_1 = -\cdot0262R_2 + 1\cdot3033G_2 - \cdot2771B_2.$

Join B_1R_2, meeting B_2G_2 in b.

$$B_1b/bR_2 = \cdot1760.$$

$$\therefore \quad B_1 = -\cdot1760R_2 + 1\cdot1760b.$$

Also $\quad B_2b/B_2G_2 = \cdot1529.$

$$\therefore \quad b = \cdot1529G_2 + \cdot8471B_2,$$

whence $\quad B_1 = -\cdot1760R_2 + \cdot1798G_2 + \cdot9962B_2.$

The above expressions contain R_2, G_2, and B_2 in T_1. units. We find the factors necessary to convert these primaries to T_2. units by finding the proportion in which they match white, as already done in the previous examples.

Dividing coefficients of R_2 by $3 \times \cdot2984$, those of G_2 by $3 \times \cdot4046$, and those of B_2 by $3 \times \cdot2970$, we obtain

$$R_1 = 1\cdot2259R_2 - \cdot222_2G_2 + \cdot1934B_2 \qquad \ldots.(a_2),$$

$$G_1 = \cdot0293R_2 + 1\cdot0738G_2 - \cdot3110_5B_2 \qquad \ldots.(b_2),$$

$$B_1 = -\cdot1966R_2 + \cdot1481G_2 + 1\cdot1181B_2 \qquad \ldots.(c_2),$$

in which all primaries are expressed in the units of their own systems. These are the equations of transformation required for conversion of data from the $R_1G_1B_1$ system to the $R_2G_2B_2$ system.

These transformation equations operate on total quantities of colours, and the result of transforming a total equation from one system gives the total equation, in the other system, of *the same absolute quantity* of the colour.

For instance, take the case already dealt with, in which the total equation of a colour in the $R_1G_1B_1$ system is

$$\cdot564C = \cdot141R_1 + \cdot056G_1 + \cdot367B_1.$$

Multiplying the coefficients on the right-hand side of (a_2) by ·141, those of (b_2) by ·056, and those of (c_2) by ·367 and collecting, we find the total amounts of R_2, G_2, and B_2 to be

$$\cdot 099R_2 + \cdot 083G_2 + \cdot 418B_2,$$

which gives, directly, the quantity on the second system, namely ·600, agreeing with the result deduced previously from the transformed unit equation and the relative values of a unit of C on the two systems.

We have seen how transformation equations such as were obtained by purely algebraical methods in the previous paper can be obtained by graphical calculation on the colour chart of one of the systems. It is obvious that we could have employed equation (7), instead of graphical calculation, in order to obtain the T_1. proportions in which R_2, G_2, and B_2 match white, and also to obtain the expressions for R_1, G_1, and B_1 in terms of T_1. units of R_2, G_2, and B_2.

This in fact gives us a non-graphical method of calculating these equations as an alternative to the purely algebraic method, and a comparison of the two will show that the numerical work is much simpler in the present case. In the former method the mathematics is applied to the general trichromatic expressions, and the various equations contain three variables. In the methods of the present paper, however, the restriction imposed by the fact that the mathematical processes, *whether carried out graphically or not*, implicitly involve the geometrical relationships of points in a plane section of the three-dimensional colour pyramid, reduces the number of variables in the various equations to two, with a corresponding simplification of the numerical work involved in their solution.

Transformation of Data between Trichromatic and Monochromatic Systems*

The system of colour specification which, for short, is usually termed the "monochromatic" system is based on the fact that any colour, except those of the purple group, can be matched by mixing monochromatic light of suitable wave-length with white. The wave-length of the monochromatic light measures the hue of the colour, and the proportion, usually expressed on a luminosity basis, of the monochromatic constituent in the matching colour is termed the "saturation." Spectral colours have therefore saturation of unity by definition.

Thus, if we represent the monochromatic constituent by H, we may define a colour by a two-constituent equation such as

$$C = sH + (1 - s)\,W \qquad \ldots\ldots(15),$$

H and W being measured in luminosity units. For colours of the purple group s is negative, H and C lying on opposite sides of W on the colour chart. It is not, therefore, possible to match C by mixing H and W, but we can match W by mixing H and C, and from such a match the value of s in equation (15) can be found. There is some diversity of practice in expressing numerical results for colours of this

* See papers by Ives already quoted; also Clara L. Froelich, *Journ. Opt. Soc. Amer.* 9 (1924) 31, and I. G. Priest, *Ibid.* 9 (1924) 503.

type. No confusion can arise if the notation of equation (15) is employed for all colours, whether s is positive or negative.

In order to convert specifications from the monochromatic system to a trichromatic system, or vice versa, it is necessary to know the positions of the spectral colours on the colour chart of the trichromatic system.

At the present time the only trichromatic system for which moderately accurate data for the spectral colours are available, is the system in which the primaries are the so-called fundamental sensations, as determined by Koenig and modified by Ives*.

It will be realized, of course, that mixture curves known for any one set of primaries are, in fact, known for any set of primaries, as, by the transformations given in this and the previous paper, the data can be converted to any desired system. There is not, however, any particular advantage for our present purpose in departing from the "Fundamental Sensation" primaries in which the Koenig-Ives data were first expressed.

Fig. 11 shows the position of the spectrum in the colour chart of this system.

Since every possible colour can be matched by positive quantities of the fundamental sensation primaries, all colour phenomena are located in one quadrant of the chart for this system. They are, in fact, all contained within the area bounded by the curved line of the spectrum and a straight line joining the two extremities of this.

If C is a colour known by its position on this chart, its hue on the monochromatic system is obtained by drawing a line from W through C to meet the spectrum in H. The wave-length at H is read off from the curve.

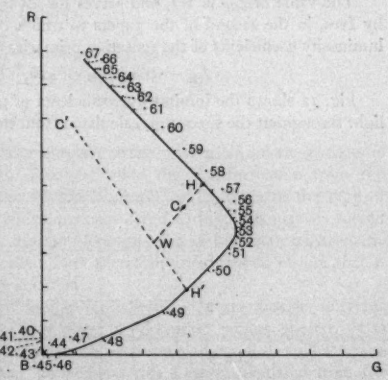

Fig. 11

The proportion of H, in the mixture of H with W which matches C, is CW/HW, where all colours are measured in trichromatic units. The saturation, in luminosity units, is therefore

$$s = \frac{CW}{HW} \times \frac{L_H}{L_C},$$

L_H and L_C being the luminosities of trichromatic units of H and C respectively,

* *Loc. cit.*

and being calculated from the luminosity coefficients of the primaries as already described.

The labour involved in calculating these coefficients is trifling; but we may nevertheless avoid the necessity of computing them for each transformation by preparing once and for all a graph showing the luminosity factors for trichromatic units of monochromatic light throughout the spectrum. Such a graph, when once prepared, enables us to read off L_H for any wave-length. Further, if instead of evaluating H/C from the colour chart, we evaluate the ratio of H to W, we avoid the necessity of computing L_G.

$$\frac{H}{W} = \frac{CW}{HC} \text{ in T. units}$$

$$= \frac{CW}{HC} \cdot \frac{L_H}{L_W} \text{ in luminosity units}$$

$$= \rho, \text{ say,}$$

whence $$s = \frac{H}{W+H} = \frac{\rho}{1+\rho}.$$

The value of L_W is 1/3, and serves for all transformations. The values given by Ives, in the second of the papers to which reference has been given, for the luminosity coefficients of the sensation primaries are

$$L_R = \cdot 568, \quad L_G = \cdot 426, \quad L_B = \cdot 006.$$

Fig. 12 shows the luminosity coefficients of unit quantities of monochromatic light throughout the spectrum, calculated from the above values.

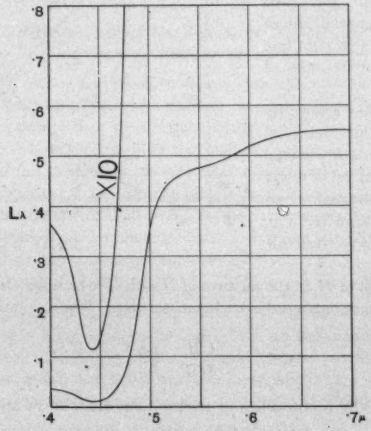

Fig. 12

The reverse transformation from the monochromatic system to the trichromatic is equally simple. Given the wave-length of the hue H and the saturation s of a colour, we first convert the ratio of H to W into trichromatic units of quantity.

$$\frac{H}{W} = \frac{s}{1-s} \text{ in luminosity units}$$

$$= \frac{s}{1-s} \cdot \frac{L_W}{L_H} \text{ in T. units.}$$

The colour therefore lies on the line joining H to W at a point such that

$$\frac{CW}{HC} = \frac{s}{1-s} \cdot \frac{L_W}{L_H}.$$

Colours such as C', in the region of the diagram (Fig. 11) to the left of the lines joining the ends of the spectrum to W, cannot be matched with monochromatic light and white, since WC', produced beyond C', does not intersect the spectrum. In this case we must produce WC' backwards to meet the spectrum at H'. By mixing C' and H' in the proportions $H'W/W'C$ (in T. units) we may match white. The ratio of H' to W in this relationship is

$$\frac{H'}{W} = -\frac{WC'}{H'C'} \text{ in T. units,}$$

$$= -\frac{WC'}{H'C'} \cdot \frac{L_H}{L_W} \text{ in luminosity units}$$

$$= \rho', \text{ say.}$$

The saturation $= \dfrac{\rho'}{1+\rho'}$, as before.

The transformation between the monochromatic and trichromatic systems is of the very greatest importance. Further, while the transformation from the monochromatic to the trichromatic system is simply an example of the type of problem in two-colour mixture which we considered earlier, and may be solved graphically or non-graphically as desired, determination of hue from trichromatic data must be performed graphically. It is therefore important to ensure that it may be carried out with adequate accuracy.

It is not, of course, sufficient to use reproductions of small scale diagrams such as appear as illustrations of this paper or the other papers quoted. It is necessary to make a proper plot of the data on a reasonably large scale so that errors of draughtsmanship are negligible. To facilitate this I have calculated from the Koenig-Ives data, as published in the second of Dr Ives' papers herein quoted, the unit equations of the spectral colours at wave-length intervals of $\cdot 002\mu$. The method of obtaining these data, while not strictly relevant to the present paper, may be of interest.

The ordinates of the spectral mixture curves tabulated in Ives' paper were converted to colour chart co-ordinates by reducing their sum to unity. This gave a series of points at intervals of $\cdot 01\mu$. These were plotted against wave-length and the best smooth line drawn through each of the three sets. With the exception of the points at $\cdot 48\mu$ and the last few points at the red end of the spectrum, the tabulated data fitted these curves accurately. The curves were finished off at the

red end so as to fit Ives' figures as well as possible while being consistent with the fact that for most observers the colour of the spectrum beyond ·67 μ is constant.

Values were then taken from the red and green curves at every ·002 μ and plotted as in Fig. 11 on a large scale colour chart. When this was done, minor irregularities in the data were revealed by points lying appreciably off the plot and by irregular variations of spacing along it. The lateral errors were first corrected and measurements were then made of the distances of the various points along the spectrum line, as measured from various suitable datum points. These were plotted in a graph, and a smooth line put through them to clean out obvious irregularities. From such curves corrected positions were found for the various wave-lengths. The location of these on the chart gave the red and green coefficients of the unit equations, the blue coefficient being then obtained by subtraction from unity. The very small values at the tail end of the blue curve beyond ·58 μ revealed, when plotted on a large scale against wave-length, some irregularities comparable with the error of reading the red and green co-ordinates from the chart. For the sake of consistency such alterations were made to the three co-ordinates as were required to give a smooth curve for the extreme end of the blue sensation.

It is not, of course, suggested that procedure such as the above improves the original data in the sense of making it more truly representative of the normal eye. It merely enables us to get rid of the minor inaccuracies inevitable in interpolation between comparatively widely spaced observations, and gives us a set of values, which are consistent among themselves to a high order of accuracy, for an eye, if such exist, which is most closely represented by the initial data.

In the fifth column of the table the value of L_λ, that is the luminosity of a trichromatic unit of monochromatic light of wave-length λ, is given for a sufficient number of wave-lengths to enable a curve such as Fig. 12 to be plotted on a suitable scale for actual use.

Wave-length in μ	a	a'	a''	L_λ	Wave-length in μ	a	a'	a''	L_λ
·390	·0560	0	·9440	·0374	·440	·0065	·0050	·9885	·0116
·400	·0540	0	·9460	·0364	2	·0043	·0070	·9887	
·402	·0534	0	·9466	·0356	4	·0028	·0090	·9882	
4	·0527	0	·9473		6	·0017	·0115	·9868	
6	·0517	0	·9483		8	·0008	·0144	·9848	
8	·0506	0	·9494		·450	·0002	·0180	·9818	·0137
·410	·0492	0	·9508	·0336	2	0	·0218	·9782	·0152
2	·0476	0	·9524		4	0	·0260	·9740	·0169
4	·0458	0	·9542		6	0	·0303	·9697	·0187
6	·0438	0	·9562		8	·0001	·0345	·9654	·0205
8	·0413	0	·9587		·460	·0004	·0412	·9584	·0234
·420	·0384	0	·9616	·0276	2	·0014	·0492	·9494	·0274
2	·0354	0	·9646		4	·0030	·0580	·9390	·0320
4	·0322	0	·9678		6	·0055	·0670	·9275	·0372
6	·0283	0	·9717		8	·0090	·0776	·9134	·0437
8	·0240	0	·9760		·470	·0134	·0891	·8975	·0509
·430	·0199	0	·9801	·0172	2	·0176	·1002	·8822	·0580
2	·0160	·0008	·9832		4	·0232	·1155	·8613	·0675
4	·0130	·0015	·9855		6	·0303	·1331	·8366	·0791
6	·0108	·0028	·9864		8	·0396	·1543	·8061	·0930
8	·0085	·0035	·9880		·480	·0512	·1805	·7683	·1106

Wave-length in μ	a	a′	a″	L_λ	Wave-length in μ	a	a′	a″	L_λ
·482	·0645	·2100	·7255	·1305	·580	·5333	·4567	·0100	·4975
4	·0779	·2426	·6795	·1516	2	·5466	·4441	·0093	
6	·0950	·2780	·6270	·1761	4	·5595	·4319	·0086	
8	·1120	·3150	·5730	·2013	6	·5735	·4186	·0079	
·490	·1250	·3580	·5170	·2266	8	·5862	·4066	·0072	
2	·1596	·3910	·4494	·2599	·590	·6000	·3935	·0065	·5084
4	·1834	·4192	·3974	·2852	2	·6135	·3804	·0061	
6	·2063	·4455	·3482	·3089	4	·6263	·3682	·0055	
8	·2286	·4695	·3019	·3316	6	·6393	·3556	·0051	
·500	·2516	·4933	·2551	·3545	8	·6515	·3438	·0047	
2	·2700	·5115	·2185	·3725	·600	·6643	·3313	·0044	·5184
4	·2840	·5245	·1915	·3858	2	·6770	·3190	·0040	
6	·2963	·5360	·1677	·3977	4	·6889	·3074	·0037	
8	·3063	·5452	·1485	·4072	6	·7007	·2959	·0034	
·510	·3175	·5550	·1275	·4175	8	·7120	·2848	·0032	
2	·3260	·5623	·1117		·610	·7225	·2745	·0030	·5273
4	·3325	·5675	·1000	·4311	2	·7306	·2667	·0027	
6	·3380	·5716	·0904		4	·7385	·2591	·0024	
8	·3462	·5764	·0774		6	·7460	·2518	·0022	
·520	·3519	·5785	·0696	·4467	8	·7530	·2450	·0020	
2	·3565	·5795	·0640		·620	·7602	·2381	·0017	·5332
4	·3611	·5800	·0589		2	·7668	·2317	·0015	
6	·3660	·5800	·0540		4	·7737	·2250	·0013	
8	·3705	·5792	·0503		6	·7804	·2185	·0011	
·530	·3762	·5780	·0458	·4602	8	·7869	·2122	·0009	
2	·3810	·5770	·0420		·630	·7930	·2063	·0007	·5383
4	·3868	·5750	·0382		2	·7995	·2000	·0005	
6	·3915	·5738	·0347		4	·8053	·1943	·0004	
8	·3960	·5719	·0321		6	·8104	·1894	·0002	
·540	·4004	·5690	·0306	·4700	8	·8154	·1845	·0001	
2	·4045	·5670	·0285		·640	·8196	·1804	·0000	·5423
4	·4095	·5640	·0265		2	·8244	·1756	0	
6	·4140	·5610	·0250		4	·8294	·1706	0	
8	·4185	·5578	·0237		6	·8345	·1655	0	
·550	·4234	·5543	·0223	·4766	8	·8397	·1603	0	
2	·4280	·5508	·0212		·650	·8441	·1559	0	·5458
4	·4325	·5470	·0205		2	·8478	·1522	0	
6	·4378	·5430	·0192		4	·8515	·1485	0	
8	·4430	·5388	·0182		6	·8550	·1450	0	
·560	·4485	·5344	·0171	·4824	8	·8577	·1423	0	
2	·4537	·5299	·0164		·660	·8610	·1390	0	·5480
4	·4600	·5243	·0157		2	·8637	·1363	0	
6	·4665	·5185	·0150		4	·8662	·1338	0	
8	·4735	·5121	·0144		6	·8689	·1311	0	
·570	·4818	·5045	·0137	·4887	8	·8711	·1289	0	
2	·4901	·4970	·0129		·670	·8731	·1269	0	·5500
4	·4999	·4879	·0122		·674	·8747	·1253	0	·5501
6	·5104	·4781	·0115		·680	·8747	·1253	0	
8	·5211	·4682	·0107		·700	·8747	·1253	0	

The number of graphical calculations which can be made on a single chart is necessarily limited. It would be a laborious process to plot the whole spectrum on every chart. I have tried the plan of working with photographic reproductions or blue prints of a colour chart on which the spectrum has been carefully plotted, but the drawback to these, particularly in the latter case, is the shrinkage of the paper on drying, which is rarely of a uniform character.

It is in comparatively few problems, however, that it is necessary, or even useful, to have the complete spectrum on the chart. For most purposes, including

transformations such as we are now discussing, we really only want, at most, a few points in the spectrum. When transforming from the monochromatic system the hue is known. We can take its co-ordinates from the above table and plot that point alone. In transforming in the reverse direction, where we have to find the hue by the intersection of a line with the spectrum, we can always tell approximately the region of the spectrum required by inspection of a chart in which the whole curve is plotted; this chart may be reserved for such reference and not used for actual graphical work. We then need only plot two or three points of the spectrum on our working chart to enable us to draw in a short length of the curve where required.

This makes it possible to perform such transformations, or any other calculations involving spectrum colours, on any piece of graph paper without the necessity of employing specially prepared charts.

The following example illustrates the simplicity of the author's procedure as applied to a transformation into the trichromatic system.

Find the fundamental sensation coefficients of a colour of hue ·580 μ *and saturation* ·4.

The proportion of H (hue) to W (white) in this colour is $\cdot 4/(1 - \cdot 4) = \frac{2}{3}$ in luminosity units. From Fig. 12 we find $L_H = \cdot 4975$, whence

$$L_W/L_H = \frac{1}{3 \times \cdot 4975} = \cdot 6700.$$

$$\therefore \quad H/W = \frac{2}{3} \times \frac{L_W}{L_H} = \cdot 4467 \text{ in T. units.}$$

The unit equation of H (from table) is

$$H = \cdot 5333R + \cdot 4567G + \cdot 0100B,$$

while

$$W = \cdot 3333R + \cdot 3333G + \cdot 3333B.$$

Hence, if the required colour is

$$C = \alpha R + \alpha' G + \alpha'' B,$$

$$\alpha = \frac{\cdot 4467 \times \cdot 5333 + \cdot 3333}{1 \cdot 4467} = \cdot 395_5, \quad \text{from equation (2)},$$

$$\alpha' = \frac{\cdot 4467 \times \cdot 4567 + \cdot 3333}{1 \cdot 4467} = \cdot 371_5,$$

and α'', either from a similar formula or by subtraction from unity, is equal to ·233.

We could, alternatively, have plotted H on the colour chart, joined HW, and found the point C such that $CW/HC = \cdot 4467 : 1$. C is the required colour and its co-ordinates give α and α' directly.

The following example illustrates the utility of these transformations in solving problems of colour mixture where data are expressed in the monochromatic system. It is intentionally couched in terms of quantities extraneous to the actual colorimetric essentials in order to illustrate the handling of problems of a practical character.

Two pieces of coloured material are placed side by side. Their colours for incident light of the quality employed are given by

	C_1	C_2
Hue	·60μ	·52μ
Saturation	·70	−1·60
Luminosity factor*	·4	·15

They are made to overlap by means of a double image prism, and are viewed through an analysing nicol set at 30° from the position in which the C_1 component of the mixture is extinguished. What is the colour of the overlapping region, and what is the field brightness, if the illumination on the specimens is 50 metre-candles?

It is necessary, before any problem involving a mixture of colours can be solved, to transform the data into the trichromatic system.

The calculation for C_1, carried out exactly as in the case illustrated in the preceding example, gives

$$C_1 = \cdot 532R + \cdot 332G + \cdot 136B.$$

The case of C_2 is treated in the same way, merely noting the negative sign attached to the saturation, and we find

$$C_2 = \cdot 317R + \cdot 125G + \cdot 558B.$$

The proportion, by luminosity, in which these colours are mixed is given by

$$C_1/C_2 = \frac{\cdot 4 \sin^2 30^\circ}{\cdot 15 \cos^2 30^\circ} = \cdot 890.$$

To convert this ratio to T. units we must calculate L_{C_1} and L_{C_2}, which we can either do in the usual way by multiplying their various coefficients by the luminosity coefficients of the primaries, or from the formula

$$L_C = \frac{mL_H + L_W}{m + 1},$$

where m is the ratio of H to W in the composition of C, measured in T. units. As we necessarily obtain both m and L_H for each of the colours C_1 and C_2 in the process of transforming from the monochromatic to the trichromatic basis, there is a slight advantage in using the latter formula. By either method we find $L_{C_1} = \cdot 444$ and $L_{C_2} = \cdot 237$. Then

$$C_1/C_2 = \cdot 890 \times \frac{L_{C_2}}{L_{C_1}} \text{ in trichromatic units} = \cdot 475.$$

Our colour mixture problem is therefore reduced to finding the colour due to mixing the colours C_1 and C_2 in the ratio ·475 : 1.

Again applying the procedure for a two-colour mixture, we find

$$C = \cdot 386R + \cdot 192G + \cdot 422B.$$

We have now to re-convert this result into the monochromatic system. Plot C

* This must not be confused with the luminosity coefficients of trichromatic units of a colour. It signifies, in the case of a filter, the ratio of the luminosities of the transmitted and incident light. In the case of an opaque material viewed by reflected light it signifies the ratio of the luminosity of the material to the luminosity of a perfectly reflecting and perfectly diffusing surface under the same illumination.

(Fig. 13) and join it to W, producing the line in the direction in which it will intersect the spectrum. A glance at Fig. 11 shows that the intersection will be in the

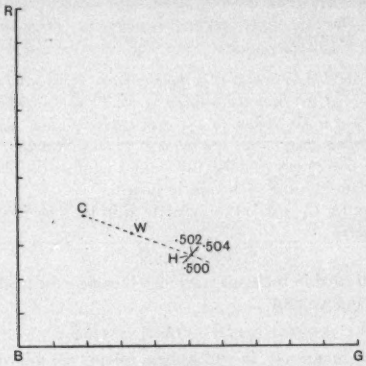

Fig. 13

neighbourhood of $\cdot 50\mu$, and a glance at the table in this neighbourhood shows that it is only necessary to plot the three points $\cdot 500\mu$, $\cdot 502\mu$, and $\cdot 504\mu$ to obtain a short length of the spectrum curve crossing the hue line. It is generally better to plot three or four points instead of only two, as a better idea is then obtained of the way the wave-length scale is varying in the neighbourhood.

The hue H is $\cdot 5018\mu$.

The proportion of H to W is

$$H/W = -\frac{WC}{HC} = -\cdot 446, \text{ in T. units}$$

$$= -\cdot 446\,\frac{L_H}{L_W} \text{ in luminosity units}$$

$$= -\cdot 446 \times \frac{\cdot 372}{\cdot 333} = -\cdot 508.$$

$$\therefore \text{ The saturation} = \frac{-\cdot 508}{1 + (-\cdot 508)} = -1\cdot 03.$$

The remaining quantitative aspects of the problem are purely arithmetical.

The light from each specimen is split by the double image prism into two beams, only one of which contributes to the image in which we are interested. In addition to this 50 per cent. reduction, C_1 is reduced in the ratio $\sin^2 30°/1 = \cdot 25$, and C_2 in the ratio $\cos^2 30°/1 = \cdot 75$. A further reduction of, say, 20 per cent. is suffered by each beam due to reflection and absorption losses in the various optical parts. The total luminosity of the field, as compared with the luminosity of a

perfectly reflecting and diffusing screen, subject to the same illumination as the specimens and observed without the intervention of any optical system, is, therefore,

$$\cdot 5 \times \cdot 80\,(\cdot 4 \sin^2 30° + \cdot 15 \cos^2 30°) = \cdot 085.$$

The actual field brightness is, therefore, that of a perfectly reflecting and diffusing screen under an illumination of $50 \times \cdot 085 = 4 \cdot 25$ metre-candles.

A transformation between trichromatic and monochromatic systems, in which, from the nature of the case, the essential points are collinear and not spread all over the chart, generally lends itself to full advantage being taken of the flexibility of the two-co-ordinate method of plotting as regards choice of scales and choice of region represented on the diagram.

Fig. 14 shows the data of Fig. 13 plotted in a way which makes much better use of the space available on a given piece of graph paper than is the case in Fig. 13.

Many cases arise in practice where, by suitable choice of scales and by confining the diagram to the region of the chart required to contain the elements of the particular problem, the accuracy of a graphical calculation can be made quite high without employing an inordinately large piece of paper.

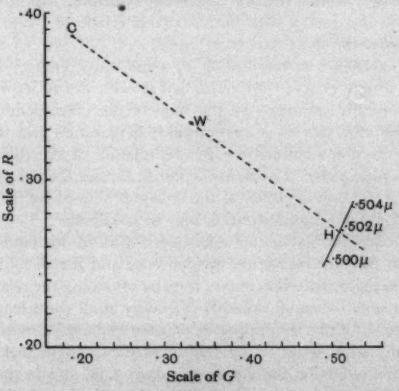

Fig. 14

As stated earlier, it is of little consequence for the majority of problems which two of the trichromatic coefficients are taken as co-ordinates of the colour chart. For general representation of data on the three-sensation basis the *R*, *G* plot adopted in this paper is definitely the best, inasmuch as it emphasizes the distinctive bends of the spectrum locus and maintains a more nearly uniform distance between it and the *W* point than either of the other plots. Further, while none of the possible methods of plotting is such that equally important colour differences are represented by equal displacements at different parts of the chart, the *R*, *G* plot is, on

the whole, better in this respect than the equilateral triangle, while either the *R*, *B* or the *G*, *B* plot is worse.

It may happen, however, that in a particular problem the points lie better for graphical solution in one of the latter plots. The difference will never be great, but in some cases where, owing to two lines intersecting at a very acute angle, the exact location of their point of intersection is difficult, the appreciable increase of inclination obtainable by plotting on the best of the three systems, whichever that may be in the particular case, may be worth obtaining.

Colour Deduced from Energy Distribution

The foregoing methods cover all problems of colour mixture in which the initial data are on the monochromatic or trichromatic systems. There is another important class of problem in which it is desired to deduce a colour from the distribution of energy in the stimulus. For this it is necessary to know the mixture curves of the spectrum for some trichromatic system, the Koenig-Ives fundamental sensation data being, as already stated, the best at present available. Mixture curves, as first obtained, refer to the energy distribution characteristic of white light, in which case the areas under the three curves are equal.

To obtain the colour corresponding to some other energy distribution, the ordinates of the mixture curves are multiplied at each of a series of suitable wave-lengths throughout the spectrum by the ratio of the corresponding ordinates of the energy distribution curves of the stimulus in question and white light. The areas of the curves now obtained for the stimulation of the three primaries are measured with a planimeter. The ratios of the *R*, *G*, and *B* areas to their sum give the coefficients of the unit equation of the colour produced by the stimulus, from which, if desired, the hue and saturation may be obtained.

It is an appreciable advantage for subsequent work of this kind if the ordinates of the white light mixture curves are divided once and for all by the ordinates of the white light energy distribution curve, thereby obtaining the relative stimulation of the three primaries for equal amounts of energy at all wave-lengths. It is then only necessary to multiply the ordinates of such reduced curves by the energy distribution in any stimulus to obtain the mixture curves for that stimulus. Ives has performed this reduction for the Koenig-Ives data, and in the second of the papers quoted mixture curves are given both for the white light and "equal energy" spectra.

The most common form in which the present problem arises is the deduction of the colour of a filter, or coloured object, from spectrophotometric data. In the general case the illuminant may not be the standard white. In such a case we multiply the ordinates of the "equal energy" mixture curves by the product of the ordinates of the energy distribution curve of the illuminant and the spectrophotometric curve of the material, which gives us the required mixture curves for the actual stimulus.

If the illuminant specified is itself the standard white, we merely have to multiply the ordinates of the *white light* mixture curves by the ordinates of the spectrophotometric curve.

To obtain the luminosity factor of the colour, for a given illuminant, the most direct method is to multiply the ordinates of the luminosity curve of the illuminant by the ordinates of the spectrophotometric curve. The ratio of the area of the reduced curve to the original is the required factor.

It is also possible to deduce the luminosity from the trichromatic specification of the colour and the luminosity coefficients of the primaries, as shown by Ives in his second paper. As described by Ives, the method is only applicable to the case in which the illuminant is white light. In other cases it is necessary to construct the mixture curves for the illuminant alone. Let the areas of these curves be A_R, A_G, and A_B. Then $A_RL_R + A_GL_G + A_BL_B$, where L_R, L_G, and L_B have their usual significance, measures, on some scale, the luminosity of the incident light. If A'_R, A'_G, and A'_B are the areas, on the same scale, of the mixture curves of the transmitted or reflected light, obtained as explained above, then the ratio

$$(A'_RL_R + A'_GL_G + A'_BL_B)/(A_RL_R + A_GL_G + A_BL_B)$$

is the required luminosity factor of the coloured material.

In such cases the luminosity curve method takes less time. A comparison of results obtained by both methods is, however, a useful criterion of the accuracy of the various data involved. In general, luminosity calculations from colorimetric data will be found less reliable than those made by the direct reduction of the luminosity curve. This is not due to any defect in the method but simply to the fact that the existing data on the relative visibility function are much more accurate than existing data concerning any system of trichromatic primaries.

When a set of mixture curves has been determined in the proper manner, with the sole intention of providing data suitable for colorimetry and not as semi-qualitative illustrations of particular theories of vision, all the methods of calculation described in this paper should give consistent results for the so-called normal eye.

Conclusion

With the methods described in the foregoing pages, any practical problem of colour mixture can be solved. It is not claimed that adequate data exist which will enable accurate and mutually consistent results to be obtained for all the types of calculation considered. The position in this respect is one which will improve as time goes on. In particular a more reliable set of spectral mixture data obtained with modern experimental precautions is required to replace the Koenig data, which, in spite of the important modifications made by Ives, give only a very approximate representation of the properties of the average eye at the blue end of the spectrum. But such practical considerations do not come within the scope of the present paper, which deals only with a system of calculation. This, in itself, is perfectly accurate, depending on no assumptions concerning the theory of vision, being, in fact, merely the mathematical representation of one initial fact of colour

mixture referred to in the opening sentences of this paper, a fact which has been common knowledge since the earliest quantitative work on colour matching was done.

As regards the relative merits of graphical and non-graphical methods of solution, where both are possible, little need be said. Both are illustrated throughout this paper, and individual workers will form their own opinions, depending partly on personal predilections and partly on available equipment. Where a calculating machine is available to facilitate numerical work, little advantage as regards time or convenience will be likely to accrue from graphical calculation.

In such circumstances the graphical method is still a valuable check for the detection of errors, arising, say, from neglect of a negative sign somewhere, which might pass unsuspected in the non-graphical calculation.

Where a calculating machine is not available the labour involved in non-graphical calculations is very much enhanced, and the graphical method is, on the whole, likely to be preferred by many people. If the work is carried out on good quality graph paper, with a fine hard pencil, using a fairly open scale, accuracy more than sufficient for all practical purposes is readily obtained. The quarter from which trouble is most likely to come—and is least likely to be suspected—is an inaccurate straight edge. Most wooden rulers, however accurate when new, tend to warp. A good steel ruler is the most satisfactory tool for the purpose.

DISCUSSION

Dr L. C. Martin: I would like to thank Mr Guild for his interesting and useful paper. It presents a clear summary of the possibilities of graphical methods and provides adequate machinery for future work on colour measurement.

I have little criticism to offer. It is not quite true that "determination of hue from trichromatic data must be performed graphically" (p. 163). It is possible by subtracting white to find the ratio of two remaining primaries and to find the wave-length of corresponding hue by interpolation from a suitable "white subtracted" table.

While it is always desirable to perfect the graphical methods as far as possible, it should not be forgotten that the accuracy of present-day colorimetry is not of a very high order; this is especially seen when attempts are made to correlate monochromatic and trichromatic analyses. Would not the inaccuracy caused by the imperfections of a reasonably good wooden straight edge be far below the probable error of experiment in 99 out of 100 cases?

While the matter does not directly concern the subject of the present paper, a certain amount of criticism has recently been made of the assumption (based on the work of Abney and Ives) that the addition of luminosities of different colours (as performed in this paper) is a valid operation. Can Mr Guild reply to this criticism? There are many other matters in this connexion—such as the limits of brightness for valid colour matches, the necessary conditions of the eyes,

the possibility of agreement between various persons—which require a vast amount of experiment. It is to be hoped that Mr Guild will be able to undertake future investigations on matters of this kind, using the methods which his present paper describes.

Mr Guild: The use of a table of two-primary ratios, as suggested by Dr Martin for the determination of hue, certainly provides an alternative to the graphical method. Whether it would be equally convenient in use one could only say after some experience of the method in practice. The suggestion is certainly an interesting one.

As regards the accuracy of present-day colorimetry, and of calculations based thereon, the accuracy is certainly not of a high order, but this does not justify inaccuracy in methods of calculation, which ought to give the correct result for the data to which they are applied. The desirable precision of a colour calculation is not determined by the magnitude of the discrepancies in existing information, but by the least change in the position of a point in the colour chart which could be detected as a change in sensation. If graphical errors are always kept less than this the calculation will give a truthful answer to the question propounded. If the question contains erroneous premises, that is quite another matter.

As regards the legitimacy of adding luminosities of lights of different spectral compositions, the objections which are from time to time advanced against this procedure appear to me to be due to lack of precise definition of the term, and to lack of knowledge of the experimental conditions under which this quantity alone, and not some mixture of this and other attributes of the total sensation, can be measured.

The experiments of Ives, described in the series of papers in the *Philosophical Magazine* during 1912, showed quite clearly that, while the comparison of luminosities in the presence of colour difference is fraught with difficulties owing to the liability of the estimate to error from a variety of psychological causes, it is nevertheless possible to eliminate these errors and to attribute to any visual sensation a *luminosity* which obeys the additive law inasmuch as the luminosity of the sensation due to any compound stimulus will measure as the sum of the luminosities of the sensations due to the various components into which the compound stimulus may be resolved, irrespective of the spectral quality of these components. He also showed that when used under specified conditions the flicker type of photometer provides a convenient method of obtaining quickly quantities which are very closely identical with luminosities as obtained with the expenditure of much time and trouble by direct comparison.

A simple example of the feasibility of adding luminosities is found in a process which is in widespread use for many important industrial and scientific purposes, namely, the calculation of the transmission factor of, say, coloured glass. This process consists of integrating the luminosities of each monochromatic constituent transmitted by the glass and assuming that the sum will give the total luminosity of the transmitted light. Such calculations always agree with the results of direct

measurement of the total luminosity to within the errors of the experimental work involved, provided the person making the direct measurement is one for whom the curve of relative visibility of radiation of different wave-lengths agrees with the visibility curve used in evaluating the luminosity of the monochromatic constituents, i.e. provided he is of normal colour vision.

The only proviso is the one which applies to all the considerations in this paper, namely, that the luminosity of the final sensation is not low enough to be within the Purkinje range or high enough to be in the dazzle range. The precise limits of luminosity which bound these regions are not at present precisely definable, but there is an enormous range—the range of ordinary visual conditions—within which the properties of the eye may be regarded as independent of brightness for practical purposes, and it is within this range that calculations such as those described in the present paper may be safely applied to the solution of problems of colour mixture.

ADDITIONS TO OUR KNOWLEDGE OF OLD SPECTACLES:

A SUMMARY OF PAPERS PUBLISHED IN 1923–24 RELATING TO THE SUBJECT OF THE THOMAS YOUNG ORATION OF 1923

By Professor MORITZ VON ROHR, Ph.D., Hon. M.D.

MS. received, 16th February, 1925. *Read and discussed, 12th March*, 1925.

ABSTRACT. Since the Thomas Young Oration of 1923 was delivered a number of important papers dealing with the history of spectacles have been published. The present article gives a summary of the fresh evidence brought forward in these papers.

As to the additional notices mentioned in the title, I shall restrict myself, as I did in the Thomas Young Oration (T.Y.O.) itself, to the period up to Young's appearance and shall not report on papers dealing with spectacles of the 19th century. I shall further arrange my report according to the chronological order of the contents, at the same time using the sectional headings of the Oration.

I. Some Remarks on the Earlier Spectacle History

Although I have not had G. Albertotti's paper(1)* in my hands, I may trust to R. Greeff's analysis of it and may mention that the author found a reliable reference to an old reading glass of 1283. We must imagine it resting on the page, as described on p. 42 of the T.Y.O. The author also gives a very early reference, possibly to a real pair of spectacles, in the year 1316, when eyeglasses with a case (in oculis de vitro cum capsula) had been bought for Bishop Arnold of Bologna.

R. Greeff(6) himself deals principally with some old German expressions for eyeglasses, "Spiegel" or "Augenspiegel" instead of the usual designation "Brille." Although his principal object is of minor interest to English readers, he gives very exact references to sources dating from the 14th to the 16th century, so that the student of spectacle history will reap material benefit even from this linguistic paper. The German designation now alone in use—"die Brille"—comes probably from the Netherlands and is, as given in T.Y.O. p. 42, a derivation from beryllus.

Of special importance are some papers(12, 13, 14, 15) by the Spanish doctor, J. P. Simon. He not only makes use of spectacle forms painted on old pictures

* The small numbers in the text refer to the list of sources and references at the end.

but also of watermarks—on paper—and of marking irons for horses, both taken from remote centuries. From the source T.Y.O. Ref. (14) I was previously aware that a collection of paper watermarks has been used for spectacle purposes by A. v. Pflugk, but here a whole collection of such marks relative to spectacle history has been systematically brought together; we learn from it beyond dispute that spectacles were in everyday use in the 14th and 15th centuries, although the writers of these remote times are remarkably reticent in mentioning these useful instruments. The different papers are written in Catalanic and it was hard work to make myself somewhat acquainted with the contents, as no Catalanic dictionary was at my disposal. In the following I shall make use of the most important items I was able to find out from these valuable papers in my restricted way. In very old times—Dr Simon reproduces one mark of 1387 but many more of the 15th century (Figs. 1 and 2)—riveted spectacles were abundant like those to be seen on the

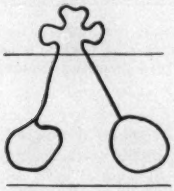

Fig. 1. Possibly a pair of spectacles with elastic pressure. Watermark of 1387.

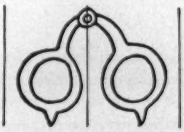

Fig. 2. Riveted spectacles with triangular prolongations. Watermark of 1423.

early Treviso picture of 1352 mentioned in T.Y.O. p. 42. They evidently could be pressed against a prominent nose in such a way that they remained sufficiently long in the desired position. If, however, the form of the face and the nose was less adapted for this simplest way of fixing the spectacles, they may have continually tumbled down; it is probable that much ingenuity was developed in order to remedy this undesirable condition. We find many of the old spectacle marks accompanied by some kind of vertical wire with one or two cross pieces (Figs. 3 and 4). Dr Simon (p. 18 H(14)) seems to look upon this as a religious emblem. Not wishing to contradict, I do not know whether it be too bald an idea to entertain that these crossed prolongations too may have been put under the head coverings, thereby giving more steadiness to the spectacles. For a certain single piece without the cross, Dr Simon (pp. 23 H and 28 H(14)) seems to entertain the same idea with regard to the head covering. At the end of the 15th century such a device was well known in Italy, as a certain sentence occurring in a sermon preached by Savonarola in 1490 has been interpreted by Italian historians as referring to this way of strengthening the hold of spectacles.

Another means for the same end, and certainly practised at that time, was the

application of elastic pressure, realized in the bow-spectacles (Fig. 5) referred to in T.Y.O. p. 44, but, whereas hitherto we could go back to the first third of the 15th century and silently assume that the spectacle mounting was made of elastic material like horn, Dr Simon has extended our knowledge in a twofold manner. He has shown

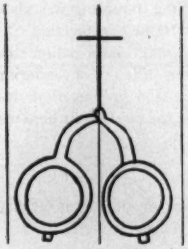

Fig. 3. Riveted spectacles with vertical wire of cross-form and quadrilateral prolongations. Watermark of 1435.

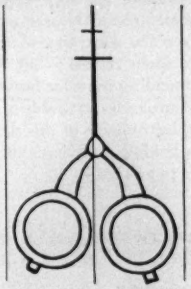

Fig. 4. Riveted spectacles with vertical wire of cross-form and quadrilateral prolongations. Watermark of 1435.

that in southern countries the elastic pressure of steel was applied and gives for it not only the watermarks 22 to 24 on p. 16 H(14) belonging to the years between 1478 and 1491 but also a very interesting reconstruction on p. 4 F(13) of such a pair of spectacles (Fig. 6) belonging to about the same time. From the reproduction we

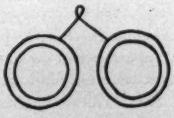

Fig. 5. Pair of spectacles with elastic pressure. Watermark of 1478.

Fig. 6. Dr Simon's reconstruction of a pair of spectacles with elastic pressure. End of 15th century.

may gather that the glasses (if not glued into their rings with mastic) were held each by a pair of metal rings, one ring being placed outwards and the other inwards. Each pair of rings was riveted together at both ends of a diameter lying in the

direction of the corresponding shank of the spring. At both lower ends, therefore, two prolongations were conspicuous, being sometimes of quadrilateral form as in Figs. 5, 6, and 7 on pp. 7 H and 8 H(14) and sometimes of triangular form as in Figs. 2, 3, 8, and 23 on pp. 6 H, 9 H, and 16 H(14).

As Dr Simon (p. 2 I(15)) pointed out, such prolongations occur also on old spectacle pictures of German artists of the 15th century; but in these cases they seem to serve as some kind of screw-holders, and the mounting was planned in a way more nearly related to our modern manner of frame construction.

The branding irons for horses did not yield so much information with regard to the old spectacles, probably in consequence of the difficulty of rendering minute details in instruments of this description. The crossed prolongations mentioned above for holding (or fixing?) the spectacles before the eyes occur here also rather often (see p. 22 H(14)).

II. The Development of the Spectacle Industry in South Germany's Free Towns during the 16th Century

Our knowledge of this period has been enlarged by a picture (Fig. 7) discovered by Prof. A. Peters(9). It was found on a single leaf (Fliegendes Blatt = fly-leaf; a forerunner to the newspapers), and the authorities do not place its date before the middle of the 17th century. The text, being of a purely symbolical nature (complaining of the increase of Brillenwesen, i.e. fraudulency and imposture), in no way explains the illustration. From the soldier's dress we may perhaps place the original of the picture as far back as the end of the 16th century, provided the picture itself is not symbolical as well. The four different sizes of the spectacle glasses are very remarkable. From the by-laws of the Regensburg spectacle makers, T.Y.O. Ref. (25), two different sizes of glasses, namely 33 and 42 mm., are known, but so great a choice as between four markedly different diameters has not been reported hitherto. The mountings seem to have been made with great care; we may suppose them to be formed of horn or leather.

IV. The Spectacle Development in Spain

I have already directed the attention of my readers in T.Y.O. p. 53 to P. Pansier's remark, Ref. (40), pointing out the early appearance of ear-spectacles. Now I think it advisable to reprint the sentence from p. 98: "On fabriqua ensuite les lunettes à branches s'attachant aux tempes ou aux oreilles. Baldinucci les signale; Peiresc nous les montre sur le nez d'un vieil espagnol." In the meantime I found in the same book on p. 55 another reference, possibly to ear-spectacles, belonging to the same collector (Fabri de Peiresc, apparently of the first third of the 17th century), and to the Iberian peninsula as well. It runs thus: "M. de Rochemaillet dit avoir observé un vieil Portugais, qui se tient chez le prince de Portugal, et qui porte continuellement à ses yeux des loupes attachées, sans

lesquelles à peine se verroit-il conduire; mais, avec cela, il voit, cognoist et lict tout de mesme qu'un autre*."

If we admit that a courtier could not very well wear spectacles mounted in leather and fastened around the head, we might here find from the expression "des loupes attachées" another instance of ear-spectacles and consider our supposition of an early Spanish invention as somewhat strengthened.

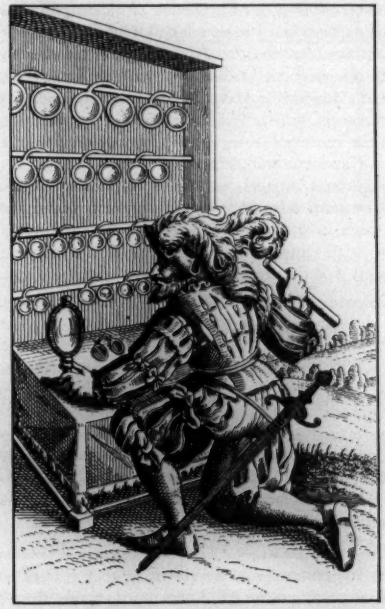

Fig. 7. Leaf of the 17th century; the picture may be satirical as the text on the fly-leaf certainly is. The soldier clothed in the garments of the 16th century is represented as looking into a crystal; taken from Ref. (9).

I was therefore disappointed at not finding any reference, at least to the former instance mentioned by Pansier, in the latest spectacle book by A. Bourgeois (3), although a whole section "Les lunettes à branches" on pp. 25–30 is given to ear-spectacles. I can only repeat my suggestion to English travellers to look up these

* Peiresc, *Manuscrits*, cahier v (1634), p. 453 (Bibliothèque de Carpentras).

places, pointed out by Pansier, during their sojourn in France. The picture of the old Spaniard with ear-spectacles is decidedly of importance for answering the question, whether the final introduction of ear-spectacles in London between 1702 and 1714 was a real invention or a successful re-invention only.

IV *a*. Old English Lenses (before 1629)

A remarkable extension of our knowledge of old English spectacles is due to Mr D. Baxandall's paper(2) and I must refer to it here, although it does not belong to the subject mentioned in the title. Its principal contents certainly treat of the telescope, but the report about William Bourne's manuscript—written probably in 1585—is of great value for the spectacle history as well, and I shall often revert to the text as given on pp. 306–7. His description of perspective glasses for seeing distant objects is the more valuable, as such instruments were at the same time known to Thomas Harriot as well, whose report was printed in 1588. On p. 306 we learn something about large bi-convex lenses made of Venetian glass, ground to the same form on both sides and worked very thin at the edge. Bourne gives a thickness of $\frac{1}{4}$ inch in the middle; if we put the thickness at the margin at $\frac{1}{10}$ inch, which is small enough for a large lens with a diameter of 1 foot, we obtain for the height p of the arc

$$p = \tfrac{1}{2}(\tfrac{1}{4} - \tfrac{1}{10}) \text{ inch} = \tfrac{3}{40} \text{ inch},$$

and by means of the easily verified formula

$$r = \frac{p^2 + h^2}{2p}$$

(h being the radius of the lens aperture = 6 inches in our case) we find

$$r = 240 \text{ ins.} = 6{\cdot}1 \text{ m.}$$

Putting $\mu = 1{\cdot}52$ for Venetian glass—hitherto we knew only that this substance was of smaller refraction than quartz, for which $\mu = 1{\cdot}54$—we obtain the power of 0·0853 D for each surface and, in consequence of the small substance in the middle, for the bi-convex lens

$$D_1 = 1/f_1' = 0{\cdot}1706 \text{ D} \quad \text{or} \quad f_1' = 5{\cdot}86 \text{ m.}$$

If we suppose a hypermetropic observer using such a weak lens and having a refractive error A of 3 D or 4 D, he must obtain a magnification M of 16·7 or 21·9 for distant objects. This may be verified by means of the formula

$$M = \frac{D_{11}}{D_{12}} = \frac{D_{11}}{D_1} \cdot \frac{A}{A + D_{11}}, \text{ where } D_{11} = 58{\cdot}6 \text{ D.}$$

These numbers for the effect of the perspective glass are much higher than the first magnifications with Dutch telescopes immediately after 1608. We must therefore not look down on this contrivance, but ought to bear in mind that as late as 1719 J. T. Desaguliers(5), a member of the Royal Society, recommended the use of a weak lens with $f_1' = 6$ ft. = 1·83 m. in order to observe the occultations of the moons of the planet Jupiter at sea with the object of ascertaining the exact

time and checking the ship's chronometer. It is difficult to say what W. Bourne could have discovered in the heavens, if he had directed his perspective glass towards them: it was not to be; this fame was to be left to Galileo's genius!

But, to revert to Bourne's description, it may have been necessary to provide some arrangement for directing the axis of this weak lens to the desired object and perhaps also some kind of tube in order to do away with false light or with reflections.

Trying to ascertain whether an arrangement similar to the perspective glass described by W. Bourne was also known in other places during this time, we shall be able to point out two other references made by contemporaries.

In 1589 the Italian I. B. Porta(11) published at Naples the second edition of his collection of curiosities of various kinds, known as his XX books on Natural Magic. In Lib. XVII, Cap. X, we find in indifferent Latin the following short paragraph:

Lente crystallina longinqua proxima videre.

Posito enim oculo in ejus centro retro lentem, remotam rem conspicator; nam quae remota fuerint, adeo propinqua videbis, ut quasi ea manu tangere videaris. Vestes, colores, hominum vultus, ut valde remotos cognoscas amicos.

We can hardly doubt that we here find the same arrangement, and, moreover, used for the same purpose as in Bourne's and in Harriot's cases. Centrum lentis, the lens centre, is to be understood as the centre of the spherical surface of the plano-convex lens. Such a form was very popular with Italian lens workers of this period, and we may safely assume it in this case also. This central point, however, is of importance with regard to the sharpness of the image only for observers with a definite hyperopia and is of no value for others. Bourne's description with his "three marvellous operacyons, or qualityes" proves, therefore, that he had made better and more elaborate experiments than his Italian contemporary.

A third evidence of nearly the same period, however, exists and is to be found in J. Kepler's book(8) of 1604, in which he laid the foundations of modern optical science. He describes his experience with an experiment made before several spectators in the technical theatre of the Electoral court at Dresden. I do not know the exact year, but it will in any case be less than 15 years after Porta's account. I now give the text from (8), p. 181:

Discus in medio crassior, seu lens crystallina, pedis diametro, stabat in egressu camerae clausae contra fenestellam, quae unica patebat, declinantem parum ad dextram. Dum igitur oculorum acies tenebrosam capacitatem pererrant, fortuito, & in locum imaginis incidunt, propiorem quidem quam lens. Cum itaque lens maligne illustraretur, oculos non admodum alliciebat. Sed neque parietes trans lentem admodum erant conspicui, quia in multis tenebris. At fenestella & circumstantes res, quae multa luce fruebantur, post lentem latentes claram sistebant in aere (me inter & lentem) sui imaginem.

We here see a similar lens to that of Bourne's—it is even possible that it had been bought in England in 1583—and the myopic Kepler had—as Bourne described it in Ref. (3) of Mr Baxandall's paper(2)—to stand farther back from the lens.

We will not treat here Kepler's interesting experiment with regard to depth localization by binocular vision but shall restrict ourselves to the statement that the perspective glass was used in Dresden in a camera clausa (apparently the same as a camera obscura). If we remember that the large camera obscura as showroom was then very much in use for entertaining gentlefolk with a demonstration of the laws of perspective (easily obtainable by these means), we may venture to bring forward our idea that the use of the perspective glass was possibly discovered in a large camera obscura, where such a weak lens had already been in regular use.

After this attempt to form a correct judgment on the large perspective glasses, we may now come to the smaller kind, which in 1585 were "commonly called spectacle glasses."

If we suppose that such a single lens ready fixed in its mounting was also known as a perspective glass, we may obtain a useful interpretation of a certain quotation in Shakespeare's *All's Well that Ends Well*, Act v, Scene iii. It runs thus:

> BERTRAM: Admiringly, my liege, at first
> I stuck my choice upon her, ere my heart
> Durst make too bold a herald of my tongue,
> Where the impression of my eye infixing,
> Contempt his scornful perspective did lend me,
> Which warped the line of every other favour,
> Scorned a fair colour, or expressed it stolen,
> Extended or contracted all proportions
> To a most hideous object.

It is perhaps possible to understand the "perspective" of the fifth line as a perspective glass of the smaller kind "commonly called spectacle glasses." They were evidently worn by distinguished persons—Bertram is a courtier—and, if somebody borrowed such a perspective glass for a moment, some alteration of the form was nearly always the consequence: according to the relation of his refractive value to the power of the temporary eyeglass he could obtain pictures enlarged or diminished and sometimes even turned upside down or "the contrary way," as Bourne has it. At the same time distortion may have been observed towards the margin of such a temporary lens held at a distance and—still more conspicuous and disgusting—an alteration in the perspective of well-known objects in the neighbourhood. Both these last effects may explain the expression "warped." (If anybody will experiment in this way, a chair or a low bookcase will serve as a good object.)

Concluding, we may point out that it seems possible, according to W. Bourne's description, that in the good society of Shakespeare's time single lenses were held as perspective glasses before one eye. If such a glass was lent to another gentleman, his eye may not have been benefited by it, but on the contrary he obtained painfully and disgustingly deformed images of the objects.

The lower classes may have used nose-spectacles (probably leather mounted),

for in *Coriolanus*, Act II, Scene ii, we find the description of the hero's joyous entrance into the capital beginning in this way:

> All tongues speak of him, and the bleared sights
> Are spectacled to see him.

Here for glasses of the crowd another expression is used and possibly denotes binocular spectacles. In this case the custom would be identical with that observed about the same time (1604) in Styria, where weak-eyed people used to wear nose-spectacles.

Der Brillenmacher.
Was hie scheint klein, wird dort gross sein.
Durch Brillen wird das kleine gross,
das Aug von Unverstand nüt los,
und seine Blödigkeit gebessert:
Des Sünders Geist im Buch der Zeit,
durchs Augen Glas der Herrlichkeit;
so wird was ewig ist, vergrossert.

Fig. 8. Pirated copy of the Dutch engraving shown in Fig. 9; by Chr. Weigel of Nürnberg 1698; taken from Ref. (10).

This supposition concerning the English manner of wearing spectacles is in good accordance with our general knowledge of the custom with regard to eye-glasses in these remote times. In the year 1560 a picture by J. van Straet shows us a man walking and holding a single glass to his eye; in 1583, according to T.Y.O. p. 47, single glasses in horn were sold in great quantities in Nürnberg; in 1623

the Spaniard Daza de Valdes expressly mentions this kind of eyeglass. Whether they were especially preferred by the nobility of that time is not known to me for certain, but we may safely assert that then nose-spectacles would have been impossible at the court of the French king.

In this way Mr Baxandall's paragraph about W. Bourne's book has led to some not improbable conclusions about English spectacle customs of this remote period; they can doubtlessly not aspire to be so certain as cases of real evidence but, as such are wanting, we must be satisfied with probabilities in order to fill up the gap in our knowledge.

Brillemaaker. 15
Ik maak Brillen, tot behulp gevonden,
Van die niet scherp meer kijken konden;
Maar warmen door de vingers niet,
Gebruikt men onze brillen niet.

Fig. 9. Stand of a Dutch spectacle vendor, after J. van Luiken 1695; taken from Ref. (7).

VI. The Growth of the Spectacle-Grinding Optician

Another historical paper by A. Peters (10) gave R. Greeff (7) the opportunity for an instructive inquiry as to the principal source of his predecessor. Whereas Peters, and others with him, had been of opinion that a copper engraving (Fig. 8) published in 1711 represented the stand of a Nürnberg spectacle vendor, Greeff could follow up its history more closely. He was able to show that it had been pirated in these times without copyright from a Dutch engraver, and that the original (of

the year 1695) in reality represented a Dutch optical stall (Fig. 9). As hitherto we did not know any details with regard to this occupation in Holland, we must be thankful for this short glimpse into Dutch optics. Nose-spectacles and terrestrial telescopes in long (cardboard) draw-tubes are not wanting, but contrary to our expectation we do not find there short perspectives, although we must suppose, from T.Y.O. p. 61, that these goods were then rather frequently made in South German towns.

The last article we have noticed with regard to our object is Hannah Breintnall's advertisement(4) of 1756 in the *Boston Evening Post*. As it is of sufficient interest, we print it here in full. Under a pair of nose-spectacles in leather (?) or horn (?) we find the following advertisement:

Just imported in the Scow Two Brothers, Cpt. Marsden, from London and to be sold by Hannah Breintnall at the Sign of the Spectacles, in Second-Street near Black-Horse-Alley

Variety of the finest chrystal spectacles, set in temple, steel, leather and other frames. Likewise true Venetian green spectacles for weak or watery eyes, of various sorts. Also concave spectacles for short sighted persons, magnifying and reading glasses, telescopes, perspectives, with multiplying glasses, and glasses for Davis's quadrants &c. &c.

The sign of the old colonial shop may be compared with Morgan's sign in T.Y.O. p. 61. As regards the text we see that the old leather spectacles have not yet been superseded by the modern steel frames. Venetian spectacles appear here with green glasses for protection against strong light. The meaning of multiplying glasses in connexion with perspectives is not clear to me.

I am indebted to Messrs Julius Springer of Berlin for the loan of the blocks for Figs. 7, 8, and 9.

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(4) Breintnall, H. [Advertisement of 1756 in the *Boston Evening Post*.] *The Opt. Journ. and Rev. of Optom.* 54 (10th July, 1924) 36.

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(7) Greeff, R. Der Brillenmacher von J. van Luiken. *Z. f. ophth. Opt.* 13 (1925) 1–4; 1 Fig.

(8) Title under J. Kepler (1) in Bin. Instr. 288–9.

(9) Peters, A. Ältere Urkunden zur Brillengeschichte. I. *Z. f. ophth. Opt.* 12 (1924) 47–50; 1 Fig.

(10) PETERS, A. Ältere Urkunden zur Brillengeschichte. II. *Ibid.* 107–10; 1 Fig.
(11) Title under I. B. PORTA (1) in Bin. Instr. 292–3.
Notes per a la historia de les ulleres pel J. P. SIMON, 8vo. (Barcelona, Impremta Badia.)
(12) SIMON, J. P. Les ulleres angulars clavades en l'escultura i pintura catalana quatrecentista. 1 E–11 E; 8 Figs.
(13) SIMON, J. P. Ulleres angulars de molla de finals del segle XV o principis del XVI. 1 F–5 F; 2 Figs.
(14) SIMON, J. P. Les ulleres en les marques del paper i en la remaderia. 1 H–28 H; 37 Figs.
(15) SIMON, J. P. Detalls referents a les llenguetes que appareixen en les ulleres angulars. 1 I–4 I; 2 Figs.

DISCUSSION

Mr D. Baxandall: I am sure that all those interested in the history of optical instruments feel very much indebted to Prof. von Rohr for providing this summary of recent contributions to our knowledge of the history of old spectacles. Its value to us is enhanced by his comprehensive and critical knowledge of the subject, and by the way he has called our attention to the more important features of the different papers to which he has referred.

I am glad that my remarks about William Bourne's perspective glass have led Prof. von Rohr towards further investigations and conclusions about English spectacle customs at the latter part of the 16th century. With reference to the computed focal length of Bourne's glass, which Prof. von Rohr gives as 19·2 feet, I may say that I took one-sixteenth of an inch as the thickness of the "very thynne" edge, and 1·53 as the refractive index of the glass. The latter value was given for yellow Venice glass by Benjamin Martin in 1759. Using these numbers, the focal length works out at 15 feet, as stated in my paper. It is not of great importance, of course, that we should know very exactly what was the focal length of this particular perspective glass. The approximate estimate—say 15 to 20 feet—which it is possible to make from the data given by Bourne is, however, of considerable value, as Prof. von Rohr points out, because such direct evidence is rarely met with at this early period.

On the question of the possible meanings which can be read into contemporary writings, I agree with Prof. von Rohr that, where more direct evidence is wanting, it is important that as much as possible of indirect evidence of this kind should be brought together so as to throw more light on the early history of spectacles and other optical instruments.

Prof. von Rohr: With regard to Mr Baxandall's remarks I beg to observe that his detection of the mean refractive index of 1·53 for Venetian glass according to B. Martin (1759) is of decided value to me, as I did not know it before. It agrees

with a more modern value of $\mu_D = 1{\cdot}53089$; $\nu = 58$ according to the Abbé Dutirou (1850).

From older writers, T. Garzoni (1585) and H. Sirturus (1618), we only learn that Venetian crystal had a lower refractive index (μ_V) than blown glass (μ_N) as used in Nürnberg and that (probably) both indices were lower than that of quartz, namely $\mu_\omega = 1{\cdot}544$. We therefore obtain

$$\mu_V < \mu_N < \mu_\omega \text{ or } 1{\cdot}53 < \mu_N < 1{\cdot}544,$$

whereas formerly I had erroneously guessed

$$\mu_V = 1{\cdot}52; \mu_N = 1{\cdot}53.$$

THE DOUBLE SQUARE: A NEW OPTICAL APPLIANCE, BASED UPON THE OPTICAL SQUARE

By J. W. GORDON, K.C.

[Communicated by Mr F. F. S. Bryson.]

MS. received, 6th April, 1925. Read and discussed, 16th April, 1925.

ABSTRACT. In this paper an optical appliance is described for which the name Double Square is suggested. The term "optical square" as used in this paper is employed in a large sense to include a pair of facing mirrors whatever the angle between them if they are used to produce a definite deflection of the reflected beam of light. The effect of the combination is in certain respects similar to that of an artificial horizon and, like that, it may be employed to determine the direction of a line of sight. When so used it supersedes the employment of webs in the focal plane of a telescope. The wide range of its usefulness is illustrated by its application to instruments so dissimilar as gun-sights, sextants, theodolites, and rangefinders.

Introduction

The optical square exhibits a particular case of the deflection of an optical axis by means of mirrors. If two mirrors are inclined to one another at any angle α and if a beam of light is reflected from the first mirror on to the second and from the second back so that its return path shall intersect the path of the incident beam, we shall have the following relation between the angle α and the divergence angle between the incident and emergent beams, calling the divergence angle x,

$$\alpha = \frac{\pi}{2} - \frac{x}{2} \qquad \ldots\ldots(1).$$

The relation is shown graphically in Fig. 1.

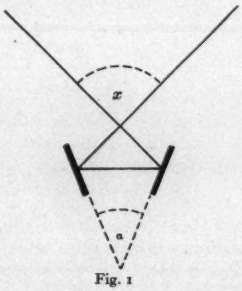

Fig. 1

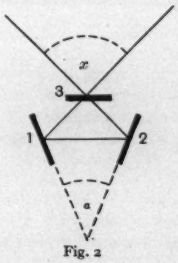

Fig. 2

Therefore we have $x = \pi - 2\alpha$ .....(2),

and putting $\alpha = \frac{\pi}{4}$ we obtain $x = \frac{\pi}{2}$, which is the well-known case of the optical square.

The more general case is the one which interests us here, and not having hit upon any more suitable name for a pair of mirrors used to produce the deviation x of our optical axis, I propose to use the term "optical square" in an enlarged sense to signify such a pair of mirrors whatever the angle α between them.

Description of Device

Taking such an optical square as one of the elements of a combination of mirrors, let us add to it another mirror reflecting the same beam, or, as the case may be, a second beam, into the observer's eye. This contrivance in its simplest form is shown in Fig. 2. Here the two mirrors 1 and 2 are placed at an angle $= \pi/4$ to one another, constituting an ordinary optical square, while the mirror 3 is placed symmetrically across the opening between the mirrors 1 and 2 and so arranged as to reflect the incident beam through the same divergence angle into the same emergent path as that into which the optical square reflects it. So far, the optical square and plane mirror act similarly upon the beam of light. But, if we substitute the one for the other, there is a marked difference between the reflected pictures, for the picture reflected by the optical square is erect, whereas that reflected by the plane mirror is inverted; that is to say, it is simply reversed, right for left or top for bottom, as the case may be. If we arrange these three mirrors as shown in Fig. 3, which presents a front view of them, a telescope having an object glass of the size of the circle in the diagram will derive image-forming rays from each of the two reflectors and will exhibit the two images in adjacent but concentric fields, the one image simply reversed in comparison with the other.

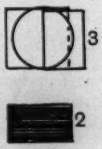

Fig. 3

Examples of Use

Artificial Horizon

Such a combination of mirrors can be used as an artificial horizon in the taking of altitudes. For this purpose the combination—the "double square"—must be levelled by disposing the surface of the plane mirror at an angle of 45° to the horizontal plane. In that position it reflects the horizon into the zenith of the instrument and if the observer's eye or, better still, a telescope be placed vertically over it, any point on the horizon may be identified by the circumstance that it falls upon the central line in which the two reversed images coincide with one another. Now, the telescope remaining unmoved in position, let the double square be rotated through some given angle. Then the optical square will continue to reflect the horizon point into the telescope but the plane mirror will so reflect some other

point—that, namely, which comes into view at the angle through which the double square has been turned. If, then, the image of this angular point is in this way brought down into coincidence with the image of the horizon, its altitude can be read, as in the use of a sextant, by a vernier travelling over a graduated arc concentric with the trunnions on which the double square turns. In this form the double square can be used to take the place of an Abney level or of a mounted sextant for observing altitudes. It would seem that it will possess over either of these instruments the great advantage that the observer will be in a position to observe either the bubble of his level or the object levelled without changing position with regard to the instrument. Furthermore, his eye will be directed downward even when he is observing the sun or a star, which may be high above his horizon, and the instrument can be so designed that this downward inclination of his optical axis shall be comfortable for prolonged observation.

Gun-Sight

A double square of this simple form may be fitted as a gun-sight, as shown in Fig. 4. In this case the double square is mounted, it may be on a director or in a turret or on a gun carriage, so that its line of sight is in a plane parallel to the axis of the gun. The target therefore will be reflected at right angles into the line of collimation of the telescope. If the telescope is removed, the double square itself defines the line of fire and may be used as an open sight. In that case it has a large

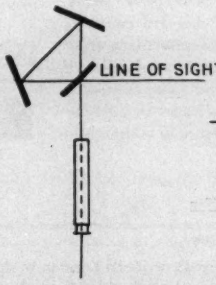

Fig. 4

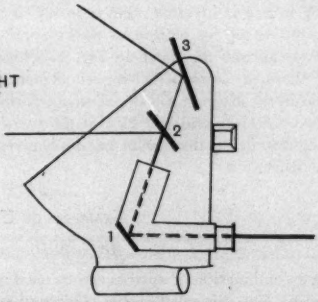

Fig. 5

field and may be used as a finder. The telescope, when brought to bear, simply increases the observer's magnifying power. It has no webs and may be of the Galilean type if the magnifying power of a Galilean telescope is sufficient for the purpose. The officer laying the gun may, upon emergency, use his binoculars or, if he uses a high-power telescope, he can mount it on a stand at any convenient distance, it may be ten or twenty feet, from the double square. As there are no webs to be illuminated, the instrument is just as available by night as by day.

It may be mentioned that in the construction of this instrument we have adopted stainless steel for the construction of the mirrors. Such mirrors are susceptible of the highest polish and of the most exact figuring. They are probably more durable than any other form of metallic mirror and they certainly have the merit of robustness. While these qualities give them especial fitness for military use, they are no disqualification for other uses and we have employed such mirrors, in some other instruments, to replace the more perishable silvered specula.

Sextant

The features described constitute notable improvements in gun-sights but even more striking are the improvements which can be made in the sextant by the adoption of the double square in its construction. Of this I have no finished model—only a diagrammatic sketch and a very rough model, the details of its design being still under discussion. In this case the optical square takes an extreme form in which the angle α is made equal to 0 and x is accordingly equal to π. The two mirrors of this "optical square" constitute the horizon glass of the sextant. The third mirror, which is made movable upon a pivot and is connected to the reading arm of the instrument, serves as its index glass. The construction is shown in Fig. 5 in which the three mirrors of the double square are numbered 1, 2, and 3, and the reading arm and graduated arc are, for the sake of compactness, disposed on the hinder side of the instrument. The handle by which it is held in use is placed at the bottom of the instrument. A steadying bar to be pressed, when the instrument is in use, against the observer's forehead, serves greatly to facilitate the taking of observations in heavy wind or a rough sea.

The instrument may be used with or without a telescope. The telescope, when a telescope is used, is of the Coudé type, being bent as shown in the diagram. The instrument, therefore, is much more compact and better balanced than a sextant of the ordinary form and, being steadied against the observer's forehead, will be found much more convenient in use.

Theodolite-Altazimuth Instrument

In all the foregoing instances the double square is of the singly reversing type, but for use in a theodolite or in an astronomical telescope, when it is to be employed for the simultaneous observation of both altitude and azimuth, the reversal must be double and coincidence in the two images juxtaposed must be limited to a single point. This result is secured by employing in place of one of the three plane mirrors a rectangular mirror which itself reverses the image which it transmits. Any one of the three mirrors may be selected for modification in this way. The model which is actually exhibited has the rectangular mirror introduced as one of the mirror surfaces of the optical square. In other respects it conforms to the gun-sight type, having its optical square constructed upon the basis of an angle α equal to $45°$. This would probably not be very convenient in use as it would involve too much bending of the observer over the instrument when the object observed lay near the

horizon. Another model of this type having an angle α equal to 30° and producing a divergence angle of 120° between the incident and emergent beams is in course of construction but not yet finished. This double square is expected, when completed, to serve as the basis of a theodolite of a very handy form and extremely comfortable in prolonged use.

Rangefinder

The double square was originally designed during the last few months of the war and with the object of serving to simplify the construction of the rangefinder, but before it could be worked out and constructed, the war had come to a close and the need of developing it in this form had ceased to be pressing. It therefore has been postponed to the simpler developments which have been above described but there seems little room for doubt that it will eventually prove as useful in connexion with the rangefinder as with the gun-sight and sextant. The reason for this opinion may be shortly stated. One of the great problems in the construction of a rangefinder is to provide effectively for the reading of the very small angle by which the range is determined. This is greatly facilitated when double squares are used for the windows of the rangefinder because the telescope can be arranged at one end of the base instead of being placed midway between the windows. It can even be placed at a distance from the rangefinder, just as in the case of the gun-sight it is placed at a distance from the gun-sight mounting, and in this way an increased radial arm can be secured on which to read this small angle. That, however, is a matter of less general interest than the application of this appliance to the instruments which have been already described, for the use of rangefinders for any but naval and military purposes is limited to a very small circle. This discussion will already have indicated how important and how numerous the applications of a double square are and for the present the matter may be left there. Its development in these and some other directions is engaging our close attention, but it is not to be supposed that, even if I entertained any ambition to treat the subject exhaustively upon the present occasion, it would be within my power to do so.

DISCUSSION

Mr T. Smith: I should like to express my appreciation of the ingenuity of Mr Gordon's sighting device. It must, however, be realized that to be successful a gun-sight must be so designed that sighting instruments may be interchanged very rapidly without affecting the line of sight and may also be set in adjustment from first principles. The arrangement here proposed seems to offer no advantages in these respects over the existing types, and indeed is likely to be more troublesome to test.

A remark may perhaps be interjected on the marking of angles between mirrors and of angles of deviation. A zero angle between two mirrors must be that for which the normals from the reflecting faces proceed in the same direction. The angle between two mirrors with their reflecting surfaces in contact, face to face, is not zero, but π or an odd multiple of π. The angle of deviation of a ray is to be taken from the incident ray *produced* to the emergent ray. Thus in Fig. 2, the light reflected at the mirror 3 is deviated through the angle $\pi - x$; the angle between the mirrors 1 and 2 is $\pi + \alpha$, so that they deviate the light through the angle $2(\pi + \alpha)$. This angle exceeds that for the single reflection by 2π, as is easily seen from the equivalence of the relation

$$2(\pi + \alpha) = 2\pi + (\pi - x)$$

to Mr Gordon's equation (2).

Mr Gordon: I fully acknowledge the systematic value of the deflection angle, but I have carefully abstained from discussing it and have substituted the *divergence angle* for the sake of having a simple angle to illustrate rather than one which is greater than π.

REMARKS ON MR CONNOLLY'S PAPER "NEW TYPES OF LEVELLING INSTRUMENTS USING REVERSIBLE BUBBLES"

By Dr A. FENNEL, Cassel

(Communicated by Dr John S. Anderson.)

MS. received, 9th February, 1925. *Read and discussed, 12th March*, 1925.

ABSTRACT. In connexion with Mr Connolly's paper on "New Types of Levelling Instruments with Reversible Levels" it is shown how an accurate test of reversible levels can be carried out in a simple manner. It is also pointed out that levelling instruments with reversible levels have been constructed in Germany since 1875.

Introduction

In his paper on "New Types of Levelling Instruments with Reversible Bubbles"* Mr Connolly has presented a clear and comprehensive discussion of the subject. He has, however, omitted to show that it is possible for the user of such a levelling instrument to satisfy himself as to the true reversibility of the bubble. Mr Weston, in the discussion which followed the paper, drew attention to the importance of testing this, and Mr Connolly suggested that it would be desirable for each bubble to be certified by a National testing institution. The author of this note, whose works have been producing levelling instruments with reversible bubbles since 1875, would like to point out a simple method of making this test.

The method to be indicated applies only to those bubbles which have the fiducial marks engraved or etched on the bubble tube itself. If this be the case, it is immaterial whether the bubble be observed direct or by means of a reflecting system. It is only when the scale is marked so as to be unalterably connected with the bubble tube that it is possible once and for all to determine the correctness of the reversion, and hence to determine the true horizontal by only two staff readings and two settings of the bubble, with a rotation of the telescope through 180°. In this connexion it should be noted that it is possible to eliminate both theoretically and practically all instrumental errors, even if the bubble tube be not fully symmetrical, provided only that the tangents at the zero points of the two scales are parallel. This leads to a somewhat different definition of the axis† of

* *Trans. Opt. Soc.* 25 (1923–24) 21.

† In Germany the "axis of the bubble" is generally understood to mean the tangent at the "zero" or central point of the scale.

the bubble from that given by Mr Connolly. In Fig. 1, *AB* and *CD* represent the inner surfaces of the two sides of the reversible bubble tube, and *E* and *F* the central points of the scales, at which points the tangents are *TT* and *T'T'* respectively. So long as these tangents are parallel it is immaterial whether *E* and *F* lie on a vertical line or not, provided that the departure from this does not exceed

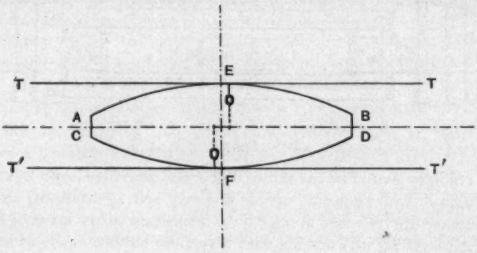

Fig. 1

a few tenths of a millimetre. But stress must be laid upon the fact that, even if the tangents at the zero points of the scales be not parallel, the bubble tube can be adjusted to the telescope axis so that the bubble will remain set when the telescope is rotated through 180° about a horizontal axis. This is explained in most German textbooks*.

Method of Test

It would be possible to check the adjustment of an instrument with a reversible bubble by the methods used in adjusting Y or dumpy levels and so deduce the correctness or otherwise of the reversible bubble. Such a procedure is, however, somewhat cumbersome and would introduce new sources of error. The most accurate and simplest method of testing a reversible bubble is that given by Amsler —the inventor of this form of bubble—in 1859, and again brought to notice by the present writer in 1892†. This test, which must be clearly distinguished from the adjusting of the instrument itself, is very simple and can be undertaken at any time on the instrument itself, if means are provided by which the bubble tube can be reversed end for end with respect to the telescope. This can easily be arranged for in the construction of the instrument, as is shown in Fig. 2. The

* Jordan, *Handbuch der Vermessungskunde*, Stuttgart (1914), vol. II, pp. 526–529; Vogler, *Lehrbuch der praktischen Geometrie*, Braunschweig (1894), p. 143; Hammer, *Lehrbuch der elementaren praktischen Geometrie*, Leipzig (1911), vol. I, pp. 674–678, etc.

† "Über eine Verbesserung von Nivellierinstrumenten mit Reversionslibelle," *Z. f. Verm. Wesen*, 21 (1892) 528–532. See also E. H. V. Melvill, "The Reversible or Double Level," *Journ. Inst. of Land-Surveyors of the Transvaal*, 2 (1910) 67–71.

mount of the bubble tube itself is fixed by two screws to a frame *R* which is adjustably attached to the telescope. Readings are taken on a staff as for the ordinary adjustment of the instrument, the bubble tube is then reversed end for end and a similar set of readings taken; if the bubble is truly reversible the means of the

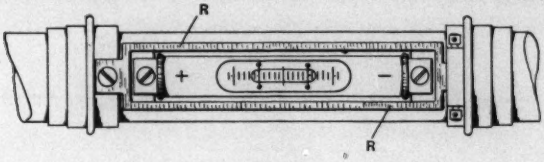

Fig. 2

two sets of readings should be the same. The error introduced by a lack of reversibility will appear with opposite sign in the two sets of readings, and the discrepancy between the two sets at once gives a measure of the error of the bubble markings. All levelling instruments with reversible bubbles made at the author's works are tested in this manner, and some 1700 bubbles have been so tested up to the end of 1924. The results of the first 706 tests are given in the *Zeitschrift für Instrumentenkunde*, 34 (1914) 54.

Some tests made in the Geodätisches Institut der Landwirtschaftlichen Hochschule in Bonn prove the permanence of the reversibility of a bubble tube*. Prof. G. Müller states that he has been unable to detect any change in this respect in the three reversible bubbles tested after a lapse of 13 years. One can therefore be confident that, when once the reversibility of the bubble has been ascertained, the adjustment of the instrument can be absolutely checked by only two staff readings and two settings of the bubble, the telescope being rotated 180° about a horizontal axis between the readings.

DISCUSSION

Mr T. F. Connolly: I have read with interest the communication from Dr Fennel. I am happy to find myself in practically complete agreement throughout. The courteous references to my work by Dr Fennel appear to be confined to the earlier paper. Many of the matters touched on have been dealt with in greater detail in my later paper on the "Theory of the Adjustment of Levels†."

With the clear demonstration as to the lack of necessity for "oppositeness" of surface curvatures I, of course, concur. Dr Fennel's Fig. 1 is practically identical with Fig. 7 in my later paper, referred to above, in which the same matter is

* *Z. f. Verm. Wesen*, 43 (1914) 113–118. See also *Kalender für Vermessungswesen und Kulturtechnik*, 41 (1912) 3–6; 44 (1915) 2.

† *Journ. Scient. Instr.* 1 (1923–24) 257, 289.

dealt with. I am glad to have the German references mentioned as I was not acquainted with them.

I am in equally full agreement as to the desirability of marking both sides and of a check on the calibration of the reversible bubbles. Two methods of check are mentioned in my later paper. The first is the workshop or laboratory test against a standard collimator and the second the field test referred to by Dr Fennel. This latter, I agree, is "cumbrous" but it is available to the user at any time without any modification to the instrument and its necessity would be rare. A third method was also used though not explicitly mentioned in my papers. The first reversible bubble level constructed for me was a Zeiss level with its bubble calibrated according to reversible bubble theory. This obviously afforded a convenient check by using the four readings of the Zeiss theory against the two readings for a calibrated reversible bubble. The theory of this check is indeed identical with that of the method suggested by Dr Fennel. The practical difference between the two consists in the replacement of an end to end reversal of the telescope by an end to end reversal of the calibrated bubble. This interesting method of check by reversal, end for end, of the bubble should be of great value to the manufacturer. I am not so sure that its incorporation in the instrument so as to be available to the user is desirable. It is not essential and it would add, though only slightly, to complexity and cost. I would prefer not to give any definite opinion on the matter at the present moment.

As an advocate of the reversible bubble I am of course delighted to have the evidence brought forward in the last paragraph of the communication as to the *permanence* of the reversibility of bubble tubes. This evidence tends to show that reversible bubbles made from good and carefully annealed glass by a first-class maker are lastingly accurate. In view of this permanence the special checking device suggested would appear to be hardly necessary.

In conclusion I would wish to express my personal pleasure in reading, and my thanks for, Dr Fennel's communication.

Major E. O. Henrici: The method given by Dr Fennel for testing the reversibility of a bubble is interesting and useful, especially to makers of such instruments, but I doubt whether it is desirable to incorporate in an actual engineer's instrument the arrangements necessary to carry it out, simple though these may be. It is not necessary for the user to be able to check this, but if he wishes to make an independent check on the adjustment the ordinary methods are open to him. In instruments where the bubble tube is enclosed in an outer case to shield it from sudden and unequal temperature changes, the complications are greater than in Dr Fennel's standard design. I feel it is desirable from every point of view to keep the design as simple as possible. I also feel that the full advantages of the reversible bubble are only obtained by using a bubble of constant length, such as has recently been introduced by Messrs E. R. Watts and Son.

Dr Fennel: I would like to thank Mr Connolly and Major Henrici for the interest they have shown in my communication. I have nothing further to add.

PHOTOMICROGRAPHS OF SECTIONS OF THE HUMAN EYE

By E. F. FINCHAM

Exhibited, 12th March, 1925.

Method of Preparation of Specimens

The section of the ciliary region is a portion of a section of the whole eye, which was cut in order to study a pathological growth in the posterior half. The anterior portion of the eye is normal. In this case the eye was fixed in formalin and after dehydration with alcohol was embedded in celloidin. Although this method retains the shape of the entire eyeball in a satisfactory manner, the preservation of the minute histological details is not perfect; neither are the sections obtained from the celloidin-embedded material sufficiently thin to allow a very high degree of microscopic resolution.

In order to show the fine detail of the retinal structure it is necessary to use a fixing agent which will give extremely good preservation of the histological elements, and also an embedding method which will allow the production of sections as thin as 5μ.

The fixing fluid which was used in this case was Kaiserling's solution, which is a mixture of formalin, potassium nitrate, and potassium acetate. Although this solution is only recommended for the preservation of museum specimens, it appears to give excellent fixation of the histological elements, and for this purpose seems to give results equal to those obtained with the expensive platino-osmic mixtures which are recommended for the fixation of the retina. The Kaiserling solution also preserves the natural colouring, which is a valuable factor in locating the macular region of the retina. When an eye which has been kept for 48 hours in this solution is opened, the macula exhibits a brilliant yellow colour, and the remainder of the retina shows the typical rose pink of the visual purple.

Kaiserling's solution does not preserve the shape of the eyeball.

For the preparation of the sections of the retina the eyes were divided after fixing and the vitreous was removed. This step is necessary to prevent the vitreous, which shrinks in the dehydrating process, from pulling the retina away from the choroid. After dehydration small pieces of retina with the choroid, and in some cases the sclerotic, attached were embedded in paraffin. The sections were cut on the Cambridge microtome.

Some difficulty was experienced in getting pieces of retina to lie sufficiently flat to cut transverse sections of the rods and cones. These sections only show a

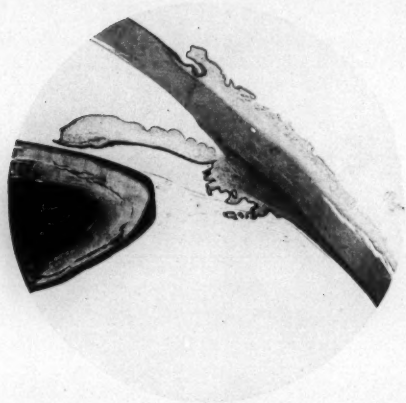

Fig. 1

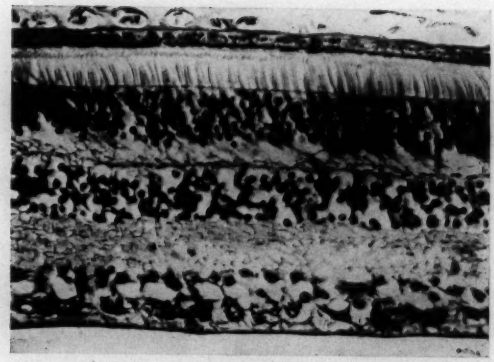

Fig. 2

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Fig. 3

Fig. 4

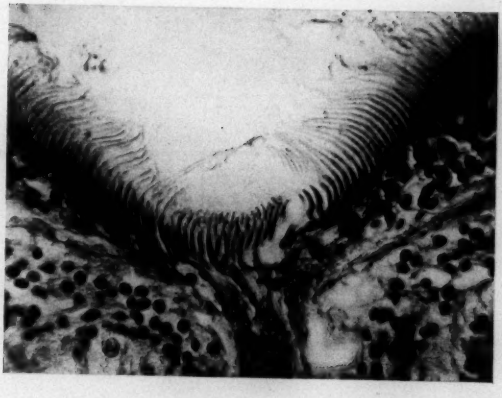

Fig. 5

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Fig. 6

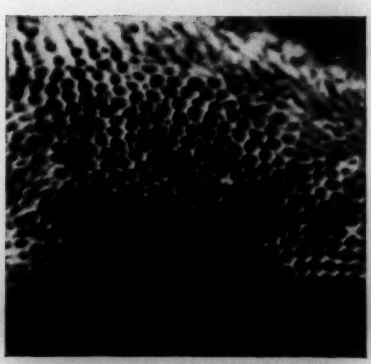


Fig. 7

very small area in each of the retinal layers, owing to the fact that the retina was not flat.

The sections of the retina were stained with acid fuchsin or with a mixture of this and orange G. Both of these dyes give beautifully sharp differentiation of cytoplasmic structures.

Description of Photomicrographs

Fig. 1. The Ciliary Region of the Eye. × 10.

This photograph shows the relative positions and the structure of the ciliary muscle, the filtration angle, and the iris. The sphincter muscle which causes the contraction of the pupil is seen as a dark band at the pupillary margin, close to the posterior surface. The section is of particular interest because it shows a complete zonule fibre, passing from the inner surface of the ciliary body to be inserted in the anterior capsule of the lens. The thickness of this fibre is 6μ. It is remarkable that the section, which is only about 12μ in thickness, should have been cut in the necessary direction to contain the entire fibre.

Fig. 2. Section of the Retina (Peripheral Portion). × 300.

The thickness of the retina at this point is 0·2 mm.; this is considerably below its maximum, which it attains at the macula.

The photograph shows at the top a portion of the choroid, in which some small vessels containing blood cells are seen in section. Below this is the layer of pigmented epithelium, the heaviest pigmentation being upon the inner surface, i.e. toward the layer of rods and cones.

The rods and cones appear to be standing upon a thin line which is the sectional view of the external limiting membrane. Below this are seen successively the outer nuclear layer, the outer synapse layer, the inner nuclear layer, the inner synapse layer, the layer of ganglion cells, and at the bottom the layer of nerve fibres and the internal limiting membrane, which in life rests in contact with the vitreous humour.

Fig. 3. The External Layers of the Retina. × 850.

This photograph shows a portion of the choroid containing blood cells, the pigmented epithelium, the rods and cones, the limiting membrane, and the external nuclear layer. The relative numbers of rods and cones show that the section is taken from a peripheral part of the retina.

Fig. 4. Transverse Section through the Bases of the Rods and Cones (Peripheral). × 550.

In this section is shown the arrangement of the rods and cones on plan. It is seen that in this part of the retina the number of rods greatly exceeds the number of cones. Their arrangement is irregular, two cones occasionally being in contact with one another.

The limiting membrane is seen filling the spaces around the cones, showing that the section is cut through the bases of these bodies. Maximum diameter of rods = 1·5μ. Maximum diameter of cones = 6μ.

Fig. 5. The Fovea. × 500.

The typical thin and elongated foveal cones are shown in this section, particularly to the right of the picture. Here the maximum diameter of the cones is 2μ. The rods are entirely absent.

Fig. 6. The External Layers of the Retina at the Border of the Fovea (Macular Region). × 900.

Here the cones exceed the rods in number. The diameter of the cones is 4·5μ.

The thinning of the nuclear layer as the fovea is approached (towards left), and the passage of the fibres in the synapse layer, away from the fovea, are shown in this section.

Fig. 7. Transverse Section of the Rods and Cones at the Edge of the Fovea. × 1000.

This section is taken from a part of the retina nearer to the centre of the fovea than that shown in Fig. 6. Only a few rods, seen as small dark spots, are present. The cones are smaller in diameter, measuring only 3μ. The hexagonal section of the cones and their packing at the fovea, as described by Helmholtz, are shown in this photograph.

REVIEWS

Die Entwicklungsjahre der Kunst, optisches Glas zu schmelzen. By Professor M. VON ROHR. Sonderdruck aus *Die Naturwissenschaften*, 39 (1924) 781–797. Pp. 17; 1 Fig. (Berlin: Julius Springer.)

In this paper Prof. von Rohr attempts a comprehensive exposition of the early history of the manufacture of optical glass from towards the end of the 18th century to the early fifties of the last century. He points out the importance of the work of Dollond and Ramsden in optical design and the shortage of good pieces of flint glass of moderate size in England at this period. This produced the proposal of the Edinburgh scientist Dr Blair (1791) to replace flint glass by liquid lenses and this idea was revived at a later time (1827) by the mathematician P. Barlow. The history of P. L. Guinand's discovery is told. Guinand's financial contract with Utzschneider and his work at Benediktbeurn under Utzschneider and later in company with J. Fraunhofer are examined. The controversial question as to the relative importance of the contributions of these three men to the development of glass melting is investigated. Prof. von Rohr appears to have made careful use of his sources and he admits that there may be French sources of information to which he has had no access. He appears to base his estimate of the importance of Fraunhofer's contribution on Abbe's and Faraday's views and these cannot be lightly set aside. Abbe stated that if the co-operation of these three men produced a greater certainty in melting glass by means of long costly operations the merit is no longer due to one but to each member of the corporation. I believe Abbe himself frequently expressed the opinion that without the assistance of the mechanic Carl Zeiss he would never probably have achieved success in his own enterprise. To the reviewer it appears that in the early stages at any rate Guinand was indispensable to the Benediktbeurn undertaking, but later on Guinand himself must have profited by contact with such a master scientific mind as that of Fraunhofer. Some light would be cast on these problems if fresh telescopes were discovered made by the Benediktbeurn works after Fraunhofer's death or by P. L. Guinand.

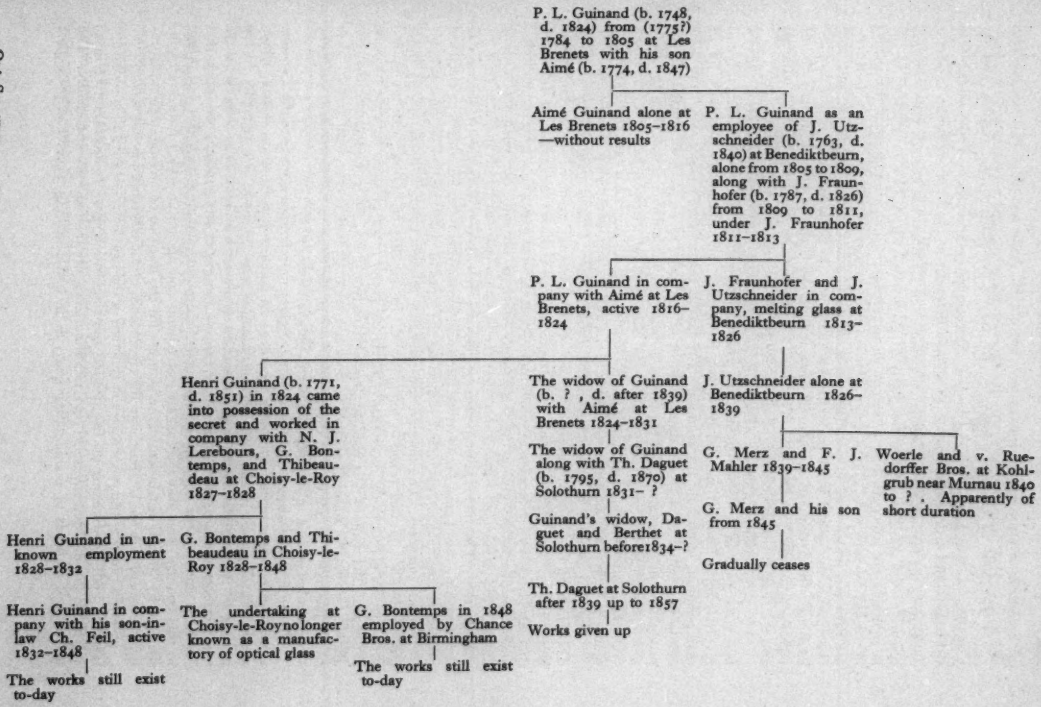
Since the Benediktbeurn factory did not supply raw optical glass but only used it for its own instruments the chief interest of English opticians centred in communications with the French and Swiss glass melters. A good deal is told of the overtures made from England to P. L. Guinand. In 1789 Parker approached him but Guinand had then no desire to communicate his secret. In the last four years of his life there seemed a possibility of his selling his secret or even of establishing himself in England. In the early fifties conditions altered. The invention of the binocular telescope and the photographic lens increased the demand for the raw material. The French and Swiss factories flourished but the Benediktbeurn works languished because they did not read the signs of the times and compete in the world market with their raw material. When G. Bontemps came to Birmingham to Chance Bros. in 1848, England was able to manufacture in competition with the Paris firms.

Such a paper as this should be of great importance to the optical glass industry in its analysis of the causes of the failures and successes of the past. After considering the lesson of this historical monograph the reviewer doubts whether any great advances in the manufacture of optical glass will be made in England without some sort of continuous State subsidy. What we should be very interested to read would be a paper by Prof. von Rohr bringing this history up to more recent times and in particular analysing the reasons for the growth and success of the optical glass industry at Jena.

A translation of Prof. von Rohr's chronological table is appended.

N. M. S. L.

Table showing the development of the art of melting optical glass.


P. L. Guinand (b. 1748, d. 1824) from (1775?) 1784 to 1805 at Les Brenets with his son Aimé (b. 1774, d. 1847)
Aimé Guinand alone at Les Brenets 1805–1816 —without results
P. L. Guinand as an employee of J. Utzschneider (b. 1763, d. 1840) at Benediktbeurn, alone from 1805 to 1809, along with J. Fraunhofer (b. 1787, d. 1826) from 1809 to 1811, under J. Fraunhofer 1811–1813
P. L. Guinand in company with Aimé at Les Brenets, active 1816–1824
J. Fraunhofer and J. Utzschneider in company, melting glass at Benediktbeurn 1813–1826
Henri Guinand (b. 1771, d. 1851) in 1824 came into possession of the secret and worked in company with N. J. Lerebours, G. Bontemps, and Thibeaudeau at Choisy-le-Roy 1827–1828
The widow of Guinand (b. ? , d. after 1839) with Aimé at Les Brenets 1824–1831
J. Utzschneider alone at Benediktbeurn 1826–1839
The widow of Guinand along with Th. Daguet (b. 1795, d. 1870) at Solothurn 1831– ?
G. Merz and F. J. Mahler 1839–1845
Woerle and v. Ruedorffer Bros. at Kohlgrub near Murnau 1840 to ? . Apparently of short duration
Guinand's widow, Daguet and Berthet at Solothurn before 1834–?
G. Merz and his son from 1845
Th. Daguet at Solothurn after 1839 up to 1857
Gradually ceases
Works given up
Henri Guinand in unknown employment 1828–1832
G. Bontemps and Thibeaudeau in Choisy-le-Roy 1828–1848
Henri Guinand in company with his son-in-law Ch. Feil, active 1832–1848
The undertaking at Choisy-le-Roy no longer known as a manufactory of optical glass
G. Bontemps in 1848 employed by Chance Bros. at Birmingham
The works still exist to-day
The works still exist to-day

A Textbook of Glass Technology. By F. W. HODKIN, B.Sc., A.I.C., and A. COUSEN, M.Sc., A.R.C.Sc., A.I.C. Pp. xxiii + 551; 251 Figs., 1 Plate. (London: Constable and Co., Ltd., 1925.) Price 42*s*. net.

During the last decade a considerable amount of attention has been directed towards research work on glass and to the study of glass technology. This is reflected in the large number of papers that have been published giving results of fundamental researches, of works' investigations, and of descriptions of processes and methods. These, however, are variously distributed in technical and scientific journals or encyclopaedias, and until the appearance of the work under review there was no book available in the English language which dealt in a comprehensive manner with the principles underlying the technology of the glass industry. The book will therefore be welcomed alike by students and teachers of the subject and by those engaged in the manufacture of glassware.

The production of such a work, dealing as it does with the properties of glass, the raw materials used in its manufacture, the chemical, physical, and mechanical problems involved in its production, and the plant, equipment, and machinery necessary for its manipulation, is no light undertaking. The task is not rendered easier when the book has to be written in such a manner as to be of use to all ranks of those engaged in the production of glassware. The expressed difficulty of the authors in deciding the limit to which each section of the subject should be elaborated can therefore be well understood. Generally the selection is good and the treatment adequate. In each case the underlying theory is sufficiently explained to permit of the various processes being understood.

The first chapter is devoted to a historical survey. This is followed by a useful discussion on the general chemical theory of the processes involved in the melting of glass batches. The "Physical Basis" of glass technology is somewhat summarily dismissed (paragraph 25) by a reference to such properties as thermal expansion, heat conductivity, and refractive index, giving the suggestion, which is certainly not what the authors intend, that physics is involved only after the glass has reached an advanced stage in its manufacture. The physical properties of glass are described in Chapter III. In view of the increasing use of glass for electrical purposes, a somewhat fuller account of its electrical properties might have been given, indicating the types of glass likely to be most suitable for various purposes. The temperature-resistance relation given in paragraph 37 holds good only for a limited range of temperature. In the chapter dealing with the optical properties of glass, which, incidentally, we should prefer to see grouped with the physical properties, the symbol η is used to denote refractive index. It is unfortunate that the almost universally adopted symbol μ does not seem to find favour with certain glass technologists. The use of η in the present case is the more to be deprecated, since the same symbol is used in the next chapter to denote viscosity.

In the section dealing with the annealing of glass, no reference is made to the Twyman annealing indicator or the annealing recorder. In connexion with strain testers, which are briefly discussed in the same section, mention may be made of a reference in the preface to the "optical methods of measuring strain, such as have been worked out in the German optical glass factories." When we remember the fundamental work that has been done on the subject by Filon and the experimental work of Coker and others in this country, and also the various instruments and apparatus which have been constructed by British instrument manufacturers for the purpose of making such measurements, we entirely fail to appreciate the significance of the reference just quoted.

Subjects such as the properties and function of the raw materials used in glass manufacture; methods of mixing the batch; fuels and combustion; gas-producers; direct-fired, gas-fired, and oil-fired furnaces; and annealing lehrs, are treated fully and clearly. The mechanical methods of manipulating glass, as in bottle making, sheet and plate-glass

making, are also fully described. While the descriptions of the various machines are useful and instructive in indicating the development of such mechanical devices, and also probably in familiarizing the student with their construction and mode of action, which, as is pointed out by the authors in the preface, is best grasped by practical inspection, it would be advantageous from an educational point of view if the subject were treated in a general manner, indicating the general principles involved in the design, construction, and operation of such machines, the individual machines being used to illustrate these principles.

In the chapter on pyrometry a description is given of all the ordinary types of pyrometers, with the exception of the disappearing-filament type, which is now well known and frequently used. Fig. 64 on p. 227 is really a diagrammatic section of the Cambridge optical pyrometer, and consequently requires the addition of another prism before an adequate description of the Wanner pyrometer can be given by reference to it. Some information might have been given on pyrometry as applied to glass making or on the method of using the instruments in connexion with glass furnaces and annealing lehrs.

The description of the making of lenses in the section dealing with optical glass is somewhat scrappy and not quite up-to-date. The machine described and illustrated is not suitable for the production of cylindrical lenses in the manner suggested. No mention is made of the moulding of lenses, of the modern disk-cutting machines, or of machines for the milling or grinding of glass with solid abrasive wheels.

Generally the book is well written, though occasionally there is a looseness of expression which is not defensible in a technological textbook, for example, "nowadays optical glasses embrace practically all other types of glass" (p. 35), "absorption (of light by glass) is dependent upon increase of temperature" (p. 37), "the further this (pin) is from the centre, the longer will be the angle of oscillation" (p. 495), while the definition of surface tension on p. 43 is too loose and inexact to be applied to any measurable quantity. There is a good index and a useful bibliography, to which might be added *Glass-making in England* by Harry J. Powell, *Notes on Glass Manufacture* by C. S. Fox (Bulletins of Indian Industries and Labour, No. 29), *The Manufacture of Optical Glass and of Optical Systems* issued by the Ordnance Department, U.S.A., and the relevant articles in Glazebrook's *Dictionary of Applied Physics*.

Such defects as may exist in the book are minor ones and do not detract greatly from the value of this laudable attempt to combine in one volume information relating to the properties of glass, the technique of its manufacture at every stage, and the principles on which such technique is based. We congratulate the authors on the success which has attended this attempt, and we recommend the book not only to students of glass technology and to those engaged in the industry, but also to others interested in the subject who desire a more intimate knowledge of the methods of manufacture of many of the articles of glass with the use of which they are already familiar.

F. F. S. B.

French-English and English-French Dictionary of Technical Terms, etc. By J. O. Kettridge, F.S.A.A., A.C.I.S. In 2 vols. or complete in 1 vol. Pp. xvi + 1137. (London: George Routledge & Sons, Ltd.) Price 50*s.* net.

The scope of this technical dictionary is "civil, mechanical, electrical and mining engineering, and allied sciences and industries, including geology, physical geography, petrology, mineralogy, crystallography, metallurgy, chemistry, physics, geometry..." and the method of arrangement is such that the author has been able to embody a method of telegraphic coding by which any entry (word or phrase) can be expressed by a ten letter cypher word.

It will be agreed that the ground covered by the above subjects is an extensive one and it is therefore a pleasant surprise to find how very thoroughly the compiler has done his work. Taking the restricted area of optics and optical engineering, a somewhat thorough test has revealed only a few omissions which might be considered as possessing any importance, viz., phare (headlight, lighthouse), papillottement (flicker), trame (web), sagittal and paraxial, with the corresponding English terms. Even of these five words the last two should, perhaps, be disregarded as alike in both languages.

The user's principal quarrel with the compiler will probably be on the score of the bulk and weight of the volume. Each word, compound of a word, hyphened or unhyphened pair of words, and each illustrative phrase is placed at the beginning of a new line, and the use of any dash or abbreviation to avoid the repetition of the main word has been carefully avoided. In his preface the compiler states that "rapid reference is rendered impossible in many works of reference by an arrangement consisting largely of paragraphic conglomerations, and by the use of signs or abbreviations, such as dashes or initial letters, to represent a recurring vocabulary word .... To begin every vocabulary entry, irrespective of its length, on a fresh line, and to repeat the vocabulary word in full whenever it recurs is the only way of escape from such confusion." This escape the compiler has certainly effected, but at the cost of more than doubling the space covered and producing a volume measuring $10 \times 7 \times 3$ inches and weighing $6\frac{1}{2}$ lbs.

No errors or misprints have been noticed, and it is evident that neither compiler nor publisher has spared any pains to produce a really accurate and trustworthy volume. The illustrative sentences liberally scattered throughout the dictionary are a particularly valuable feature, and there is no doubt whatever that the book will be of very great value to anyone whose work demands the use of a dictionary of this kind.

J. W. T. W.

Fremdsprachliches Optisches Wörterbuch (*deutsch-englisch-französisch-spanisch*). *Teil I: Das Auge*. Herausgegeben von der Deutschen Optischen Wochenschrift. Bearbeitet von Dr W. Pietzsch. Pp. viii + 93. (Erfurt: Gebr. Richters Verlagsanstalt.) Price G.M. 5.

This is the first part of an optical vocabulary edited by Dr W. Pietzsch of Jena, and published under the auspices of the Deutsche Optische Wochenschrift. The other six parts are to deal with spectacles, magnifying glasses, microscopes, field and opera glasses, astronomical instruments, and photographic optics. The terms are arranged alphabetically in German, and the corresponding English, French, and Spanish terms are then given in parallel columns. For convenience in passing from any one of these languages to German, an index is given for each language and in this the terms are arranged alphabetically in that language and each is followed by a reference to the page and line where this term and its equivalents are to be found in the body of the book. Ten pages at the end are devoted to a list of the titles, and particulars of publication, of the principal optical journals published throughout the world.

The general arrangement is good and the type clear and well displayed. The ground covered is, in general, adequate but there are several terms which might have been expected to be included, such as Sehpurpur, Nachbild, Eintritts- u. Austrittspupille. It does not appear as if the vocabulary had been prepared by searching thoroughly the principal standard works on the subject. It is, nevertheless, a valuable addition to the reference library of the practical optician.

J. W. T. W.

ABSTRACTS OF PATENT SPECIFICATIONS

219964. Zeiss, C. LENSES.

A high-aperture lens for micro-projection purposes consists of five elements. Two collective lenses, separated by a dispersive element, are followed by a cemented doublet, of which the outer element is a dispersive meniscus.

220121. Denne, M. T. MICROSCOPES.

The main limb of the instrument carries, together with the lens system, an optical bench for use with sub-stage apparatus. The whole is movable upon a horizontal axis, in a carrier secured to a three-pillar stand.

220173. Beck, C., and others. MICRO-PROJECTION APPARATUS.

A self-contained instrument for micrometric work includes an illuminating system and an object-holder placed before a microscope which projects into the lower part of a casing. A mirror within the casing throws the image upon a screen, which is provided with a scale graduated according to the magnification of the system.

220266. Atha and Co. MAGNIFYING LENSES.

A magnifying lens comprises a front dispersive lens and a back converging lens, simple or compound, the space between them forming a collective air lens.

220848. Heath, G. A. BINOCULARS.

The eyepieces are adjustable in sockets which are themselves operated by a central focussing screw. A scale of graduations is provided for each movement.

220961. British Thomson Houston Co., Ltd. LENSES.

Condenser lenses are constructed to throw a number of overlapping images of the source upon the gate, so as to produce a more even illumination. Portions of the lens along two planes at right angles may be removed, or a composite lens may be employed to produce the same effect.

221415. Pattison, E. W. BI-FOCAL LENSES.

A small insert is fused on to a flat formed upon a main blank of lower refractive index, and the composite disk is then ground to the desired base curve. The curvature of the back surface is left at disposal.

221536. Culver, Ltd., G., and another. LENS MOUNTS.

Two bevelled rings, placed face to face, provide a groove in which the lens is held, a band placed around the rings then securing them together.

221764. Leitz, E. MICROSCOPIC CONDENSERS.

The full aperture of the objective is rendered effective, in dark ground illumination, by so arranging the reflecting condenser that the delimitation of the light is rendered operative in a surface which is conjugated to the aperture stop of the objective.

223476. Kakabadzé, D. STEREOSCOPIC PROJECTION.

The right and left images are projected successively upon the screen by an oscillating mirror, or a pair of shutters, the respective light paths being widely separated to enable the eye to pick up the proper image. To avoid the distortion due to oblique projection, the axes of the objectives may be inclined to the path of the light.

223837. General Electric Co., Ltd. FUSIBLE GLASS.

Glasses or enamels of low melting point consist of a lead borate or boro-silicate, coloured by means of cobalt or other material. They are used as an adhesive between metal, glass, or porcelain, or for providing a marking which may be burnt on at a relatively low temperature.

224258. Connolly, T. F. SURVEYING INSTRUMENTS.

In instruments in which a reversible bubble-tube is mounted upon a rotatable telescope, the tube is carried in bearings which are adjusted until rotation of the tube about its axis produces no displacement of the bubble relative to the central plane.

224393. Pictet, L., and another. STEREOSCOPIC PROJECTION.

Stereoscopic images obtained from simultaneous exposures are projected in superposition by light polarized in different planes, observation being made through a pair of analysers.

224425. Lee, H. W., and another. LENSES.

Numerical data are given for an objective consisting of four simple components, a leading collective lens being followed by a dispersive component, and then by a pair of components slightly separated from each other.

225042. Barr and Stroud and another. RANGEFINDERS.

An adjustable prism system is provided to deflect a portion of the beam and form a supplementary image. The displacement necessary to establish coincidence between two dissimilar parts in the two images gives a reading for the angular subtense of the distance between these parts, and upon an assumption as to its actual value, a reading of the range is obtained.

225282. Ryland, H. S. PROJECTORS.

A concave spherical reflector faces a larger spherical mirror of annular shape, having its centre of curvature at about the focus of the first. The source of light is preferably placed a short distance behind this point, and the light emerges through the centre of the annular mirror.

225365. Davis, A. L. PROJECTORS.

Consists of a light projector for use in surveying by air-craft. The lamp is partly countersunk into the shell of the machine, and is arranged so that the reflector can be drawn back and tilted with respect to the bulb, which is itself adjustable in various ways.

225398. Hill, R., and others. WIDE-ANGLE LENSES.

A divergent meniscus with a convex front surface and a steep concave back face is used in conjunction with a positive system. The image is approximately stereographic.

225412. Alldridge, R. S., and another. MICROSCOPES.

The fine adjustment is made by a cone, moved by a micrometer screw, and formed with a thread upon its surface to engage one end of a lever, the other end of which supports and moves the tube of the instrument.

225556 *. Chrétien, H. REFLECTORS.

A lens system, preferably achromatic and corrected for spherical aberration and coma, is mounted in front of a reflector, the back surface of which is curved so as to coincide with the curvature of the image produced by the anterior elements. Incident light is thus returned upon its own path. Elements so related may be mounted upon the two faces of a sheet of glass, to provide optical indications of various kinds.

225690. Waldram, J. M. PHOTOMETERS.

In a comparison photometer, the light from one source is arranged to provide a graduated luminous background, while the other furnishes two separate patches, which are seen against it. Adjustments are made until one patch appears brighter, and the other darker, than the background at those points.

226594. Burns, W. A. SKY SIGNS.

A transparent reflector, placed at an angle, is employed to form an image of an illuminated device in such a way as to produce an illuminated sign, apparently unframed and unsupported in space, against the normal background of sky or roofs.

226697. Beck, C., and another. MICROSCOPES.

Reflectors for opaque objects, mounted between the object glass and the eyepiece, are provided with two tilting adjustments, in planes at right angles to each other, and with two sliding adjustments at right angles to the axis of the instrument.

226838 *. Marsat, A. J. B. VEHICLE LAMPS.

The bulb is fitted with a transverse horizontal filament, placed at the focal point of a long focus lens, which forms the front of the lamp. The body consists of a deep paraboloid, the upper part of which is reflecting or diffusing, and the lower part diffusing or absorbing. Two part-spherical mirrors may be placed behind the bulb.

227072. Hilger Ltd., A., and another. COLOUR PROJECTORS.

The appropriate shades to be used in a design or picture are determined by producing a number of positive transparencies, each corresponding to a single shade in the original, and projecting them simultaneously from an apparatus in which the colour and luminosities can be varied, until a satisfactory blend is arrived at.

R. J. T.

* Specification published before acceptance.

ERRATA

Vol. XXVI, No. 1, Mr T. Smith's paper, "The Back Vertex Power of a Combination of Lenses." The last equation on p. 37 should read

$$F = \frac{V - B}{1 + kV(V - B)}.$$

Vol. XXVI, No. 2, p. 117, review of *Helmholtz's Treatise on Physiological Optics*, Vol. 1. The statement that the book is not on sale in this country is not quite correct. Copies may be obtained from

THE EDITOR,
THE OPTICIAN,
72–78, FLEET STREET, E.C. 4.

SIR WILLIAM HERSCHEL, F.R.S.

from a medallion, by W. Tassie, in the possession of Prof. F. J. Cheshire.